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FUNDAMENTAL SOLUTION IN THE FULLY COUPLED THEORY OF ELASTICITY FOR SOLIDS WITH DOUBLE POROSITY

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Abstract. In this paper the 2D fully coupled quasi-static theory of poroelasticity for materials with double porosity is considered. For these equations the fundamental and some other matrixes of singular solutions are constructed in terms of elementary functions. The properties of single and double layer potentials are studied.

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Introduction

The theory of consolidation for elastic materials with double porosity was presented in [1-3]. The theory of Aifantis unifies the models of Barenblatt for porous media with double porosity [4] and Biot's model for porous media with single porosity [5]. However, Aifantis' quasi-static theory ignored the cross-coupling effects between the volume change of the pores and fissures in the system. This deficiency was eliminated and cross-coupled terms were included in the equations of conservation of mass for the pore and fissure fluid and in Darcy's law for solid with double porosity in [6]. In [6,7] the cross-coupled terms were included in Darcy's law for solid with double porosity.

The double porosity concept was extended for multiple porosity media in [8, 9]. The basic equations of the thermo-hydro-mechanical coupling theory for elastic materials with double porosity were presented in [10-12]. The theory of multiporous media, as originally developed for the mechanics of naturally fractured reservoirs, has found applications in blood perfusion. The double porosity model would consider the bone fluid pressure in the vascular porosity and the bone fluid pressure in the lacunar-canalicular porosity. An extensive review of the results in the theory of bone poroelasticity can be found in the survey papers [13-15]. For a history of developments and a review of main results in the theory of porous media see [16].

The fundamental solutions have occupied a special place in the theory of PDEs. They are encountered in many mathematical, mechanical, physical and engineering applications. Indeed, the application of fundamental solutions to a recently developed area of boundary element methods has provided a distinct advantage in the fact that an integral representation of solution of a boundary value problem by fundamental solution is often more easily solved by numerical methods than a differential equation with specified boundary and initial conditions. Recent advances in the area of boundary element methods, where the theory of fundamental solutions plays a pivotal role, has provided a prominent place in research of problems in the theories of PDEs, applied mathematics, continuum mechanics and quantum physics. The fundamental solutions in the linear theories of elasticity and thermoelasticity for materials with microstructures are constructed by means of elementary functions by several authors [17-20].

In this paper the 2D fully coupled quasi-static theory of poroelasticity for materials with double porosity is considered. For these equations the fundamental and some other matrixes of singular solutions are constructed in terms of elementary functions. The properties of single and double layer potentials are studied.

2. Basic equations

Let $\mathbf{x} = (x_1, x_2)$ be a point of the Euclidean 2D space E^2. Let D^+ be a bounded 2D domain surrounded by the curve S and let D^- be the complement of $D^+ \cup S$. $D_x = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)$. Let us assume that the domain D is filled with an isotropic material with double porosity.

The system of homogeneous equations in the 2D fully coupled quasi-static linear theory of elasticity for solids with double porosity can be written as follows

$$\begin{gathered}\mu\Delta\mathbf{u} + (\lambda+\mu) graddiv\mathbf{u} - grad(\beta_1 p_1 + \beta_2 p_2) = 0,\\ i\omega\beta_1 div\mathbf{u} + (k_1\Delta + a_1)p_1 + (k_{12}\Delta + a_{12})p_2 = 0,\\ i\omega\beta_2 div\mathbf{u} + (k_{21}\Delta + a_{21})p_1 + (k_2\Delta + a_2)p_2 = 0,\end{gathered} \tag{1}$$

where $\mathbf{u} = (u_1, u_2)^T$ is the displacement vector in a solid, p_1 and p_2 are the pore and fissure fluid pressures respectively. $a_j = i\omega\alpha_j - \gamma, a_{ij} = i\omega\alpha_{ij} + \gamma$, $\omega > 0$ is the oscillation frequency, β_1 and β_2 are the effective stress parameters, $\gamma > 0$ is the internal transport coefficient and corresponds to fluid transfer rate with respect to the intensity of flow between the pore and fissures, α_1 and α_2 measure the compressibilities of the pore and fissure system, respectively; α_{12} and α_{21} are the cross-coupling compressibility for fluid flow at the interface between the two-pore systems at a microscopic level. λ, μ, are constitutive coefficients, $k_j = \frac{\kappa_j}{\mu'}$, $k_{12} = \frac{\kappa_{12}}{\mu'}$, $k_{21} = \frac{\kappa_{21}}{\mu'}$, μ' is the fluid viscosity, κ_1 and κ_2 are the macroscopic intrinsic permeabilities associated with matrix and fissure porosity, respectively, κ_{12} and κ_{21} are the cross-coupling permeabilities for fluid flow at the interface beetween the matrix and fissure phases, Δ is the Laplacian. The superscript "T" denotes transposition.

We assume that the inertial energy density of solid with double porosity is a positive definite quadratic form. Thus, the constitutive coefficients satisfy the conditions

$$\mu > 0, \quad k_1 > 0, \quad a_1 a_2 > a_{12} a_{21}, \quad k_1 k_2 > k_{12} k_{21}, \quad \gamma > 0.$$

We introduce the matrix differential operator with constant coefficients:

$$\mathbf{A}(D_x, \omega) = (A_{ij})_{4\times 4},$$

where

$$A_{lj} := \delta_{lj}\mu\Delta + (\lambda+\mu)\frac{\partial^2}{\partial x_l \partial x_j}, \quad l,j=1,2,$$
$$A_{j3} := -\beta_1\frac{\partial}{\partial x_j}, \quad A_{j4} := -\beta_2\frac{\partial}{\partial x_j}, \quad j=1,2,$$
$$A_{3j} := i\omega\beta_1\frac{\partial}{\partial x_j}, \quad A_{4j} := i\omega\beta_2\frac{\partial}{\partial x_j}, \quad j=1,2, \quad A_{33} := k_1\Delta + a_1,$$
$$A_{34} := k_{12}\Delta + a_{12}, \quad A_{43} := k_{21}\Delta + a_{21}, \quad A_{44} := k_2\Delta + a_2,$$

δ_{lj} is the Kronecker delta. Then the system (1) can be rewritten as

$$\mathbf{A}(D_x,\omega)\mathbf{U} = 0, \tag{2}$$

where

$$\mathbf{U} := (\mathbf{u}, p_1, p_2).$$

The conjugate system of the equation (1) is

$$\begin{aligned} &\mu\Delta\mathbf{u} + (\lambda+\mu) grad div\mathbf{u} - i\omega grad(\beta_1 p_1 + \beta_2 p_2) = 0,\\ &\beta_1 div\mathbf{u} + (k_1\Delta + a_1)p_1 + (k_{21}\Delta + a_{21})p_2 = 0,\\ &\beta_2 div\mathbf{u} + (k_{12}\Delta + a_{12})p_1 + (k_2\Delta + a_2)p_2 = 0,\end{aligned} \tag{3}$$

$$\widetilde{\mathbf{A}}(D_x,\omega)\mathbf{U} = \mathbf{A}^T(-D_x,\omega)\mathbf{U} = 0.$$

We assume that $\mu\mu_0(k_1k_2 - k_{12}k_{21}) \neq 0$, where $\mu_0 := \lambda + 2\mu$. Obviously, if the last condition is satisfied, then $\mathbf{A}(D_x,\omega)$ is the elliptic differential operator.

3. The basic fundamental matrix

In this section, we will construct the basic fundamental matrix of system (2). We introduce the matrix differential operator $\mathbf{B}(\partial\mathbf{x})$ consisting of cofactors of elements of the matrix $\mathbf{A}^T$ divided on $\mu\mu_0(k_1k_2 - k_{12}k_{21})$:

$$\mathbf{B}(D_x) = (B_{ij})_{4\times 4},$$

where

$$B_{lj} = \frac{\delta_{lj}}{\mu}\Delta(\Delta+\lambda_1^2)(\Delta+\lambda_2^2) - \xi_l\xi_j\frac{i\omega}{\alpha_0}(\alpha_{12}+\alpha_{11}\Delta)$$
$$-\xi_l\xi_j\frac{\lambda+\mu}{\alpha_0}[(k_1k_2 - k_{12}k_{21})\Delta\Delta + k_0\Delta + a_1a_2 - a_{12}a_{21}],$$

$$B_{3j} = -\frac{i\omega\mu}{\alpha_0}\xi_j\Delta[(\beta_1 k_2 - \beta_2 k_{12})\Delta + \beta_1 a_2 - \beta_2 a_{12}],$$
$$B_{4j} = -\frac{i\omega\mu}{\alpha_0}\xi_j\Delta[(\beta_1 k_{21} - \beta_2 k_1)\Delta + \beta_1 a_{21} - \beta_2 a_1],$$
$$B_{j4} = -\frac{\mu}{\alpha_0}\xi_j\Delta[(\beta_1 k_{12} - \beta_2 k_1)\Delta + \beta_1 a_{12} - \beta_2 a_1],$$
$$B_{j3} = \frac{\mu}{\alpha_0}\xi_j\Delta[(\beta_1 k_2 - \beta_2 k_{21})\Delta + \beta_1 a_2 - \beta_2 a_{21}], \quad \xi_j = \frac{\partial}{\partial x_j}, \quad l, j = 1, 2,$$
$$B_{33} = \frac{\mu}{\alpha_0}\Delta\Delta[\mu_0 k_2\Delta + \mu_0 a_2 + i\omega\beta_2^2], \quad B_{44} = \frac{\mu}{\alpha_0}\Delta\Delta[\mu_0 k_1\Delta + \mu_0 a_1 + i\omega\beta_1^2],$$
$$B_{43} = -\frac{\mu}{\alpha_0}\Delta\Delta[\mu_0 k_{21}\Delta + \mu_0 a_{21} + i\omega\beta_1\beta_2], \quad B_{34} = -\frac{\mu}{\alpha_0}\Delta\Delta[\mu_0 k_{12}\Delta + \mu_0 a_{12} + i\omega\beta_1\beta_2],$$
$$k_0 = a_1 k_2 + a_1 k_1 - k_{12}a_{21} - k_{21}a_{12}, \quad \mu_0 = \lambda + 2\mu, \quad \alpha_0 = \mu\mu_0(k_1 k_2 - k_{12}k_{21}),$$

δ_{lj} is the Kronecker delta.

Substituting the vector $\mathbf{U}(\mathbf{x}) = \mathbf{B}(\partial\mathbf{x})\mathbf{\Psi}$ into (1), where $\mathbf{\Psi}$ is a four-component vector function, we get

$$\Delta\Delta(\Delta + \lambda_1^2)(\Delta + \lambda_2^2)\mathbf{\Psi} = 0,$$

λ_j^2 are roots of equation

$$\mu_0(k_1 k_2 - k_{12}k_{21})\xi^2 - (\mu_0 k_0 + i\omega\alpha_{11})\xi + \mu_0(a_1 a_2 - a_{12}a_{21}) + i\omega\alpha_{12} = 0, \tag{4}$$

$$\alpha_{11} = k_2\beta_1^2 + k_1\beta_2^2 - \beta_1\beta_2(k_{12} + k_{21}),$$
$$\alpha_{12} = a_2\beta_1^2 + a_1\beta_2^2 - \beta_1\beta_2(a_{12} + a_{21}).$$

Whence, after some calculations, the function Ψ can be represented as

$$\Psi = \frac{r^2(\ln r - 1)}{4\lambda_1^2\lambda_2^2} - \frac{1}{\lambda_1^2 - \lambda_2^2}\left[\frac{\varphi_1 - \ln r}{\lambda_1^4} - \frac{\varphi_2 - \ln r}{\lambda_2^4}\right], \tag{5}$$

where

$$\varphi_m = \frac{\pi}{2i}H_0^{(1)}(\lambda_m r),$$

$H_0^{(1)}(\lambda_m r)$ is Hankel's function of the first kind with the index 0

$$H_0^{(1)}(\lambda_m r) = \frac{2i}{\pi}J_0(\lambda_m r)\ln r + \frac{2i}{\pi}\left(\ln\frac{\lambda_m}{2} + C - \frac{i\pi}{2}\right)J_0(\lambda_m r)$$

$$-\frac{2i}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{(k!)^2}\left(\frac{\lambda_m r}{2}\right)^{2k}\left(\frac{1}{k} + \frac{1}{k-1} + ... + 1\right),$$

$$J_0(\lambda_m r) = \sum_{k=0}^{\infty}\frac{(-1)^k}{(k!)^2}\left(\frac{\lambda_m r}{2}\right)^{2k}, \quad r^2 = (x_1 - y_1)^2 + (x_2 - y_2)^2, \;\; m = 1, 2.$$

Substituting (5) into $\mathbf{U} = \mathbf{B\Psi}$, we obtain the matrix of fundamental solutions for the equation (1) which we denote by $\mathbf{\Gamma}(\mathbf{x}\text{-}\mathbf{y})$

$$\mathbf{\Gamma}(\mathbf{x}\text{-}\mathbf{y}) = \| \Gamma_{kj}(\mathbf{x}\text{-}\mathbf{y}) \|_{4\times4},$$

where

$$\Gamma_{kj}(\mathbf{x}\text{-}\mathbf{y}) = \frac{\ln r}{\mu}\delta_{kj} - \frac{\partial^2 \Psi_{11}}{\partial x_k \partial x_j}, \quad \Gamma_{j3}(\mathbf{x}\text{-}\mathbf{y}) = \frac{\partial \Psi_{13}}{\partial x_j}, \quad k, j = 1, 2,$$

$$\Gamma_{j4}(\mathbf{x}\text{-}\mathbf{y}) = -\frac{\partial \Psi_{14},}{\partial x_j}, \quad \Gamma_{3j}(\mathbf{x}\text{-}\mathbf{y}) = -\frac{\partial \Psi_{31}}{\partial x_j}, \quad \Gamma_{4j}(\mathbf{x}\text{-}\mathbf{y}) = \frac{\partial \Psi_{41}}{\partial x_j},$$

$$\Gamma_{33}(\mathbf{x}\text{-}\mathbf{y}) = \frac{\mu}{\alpha_0(\lambda_1^2 - \lambda_2^2)}\left[m_{12}\varphi_2 - m_{11}\varphi_1\right], \quad m_{1j} = -\mu_0 k_2 \lambda_j^2 + \mu_0 a_2 + i\omega\beta_2^2,$$

$$\Gamma_{44}(\mathbf{x}\text{-}\mathbf{y}) = \frac{\mu}{\alpha_0(\lambda_1^2 - \lambda_2^2)}\left[m_{22}\varphi_2 - m_{21}\varphi_1\right], \quad m_{2j} = -\mu_0 k_1 \lambda_j^2 + \mu_0 a_1 + i\omega\beta_1^2,$$

$$\Gamma_{34}(\mathbf{x}\text{-}\mathbf{y}) = \frac{-\mu}{\alpha_0(\lambda_1^2 - \lambda_2^2)}\left[n_{12}\varphi_2 - n_{11}\varphi_1\right], \quad n_{1j} = -\mu_0 k_{12} \lambda_j^2 + \mu_0 a_{12} + i\omega\beta_1\beta_2,$$

$$\Gamma_{43}(\mathbf{x}\text{-}\mathbf{y}) = \frac{-\mu}{\alpha_0(\lambda_1^2 - \lambda_2^2)}\left[n_{22}\varphi_2 - n_{21}\varphi_1\right], \quad n_{2j} = -\mu_0 k_{21} \lambda_j^2 + \mu_0 a_{21} + i\omega\beta_1\beta_2,$$

$$j = 1, 2, \quad \Psi_{11} = \left[(\lambda + \mu)(a_1 a_2 - a_{12} a_{21}) + i\omega\alpha_{12}\right] \frac{r^2(\ln r - 1)}{4\alpha_0 \lambda_1^2 \lambda_2^2},$$

$$+ \frac{i\omega\mu}{\mu_0 \alpha_0 (\lambda_1^2 - \lambda_2^2)} \sum_1^2 (-1)^j \left(\alpha_{11} - \frac{\alpha_{12}}{\lambda_j^2}\right) \frac{\varphi_j - \ln r}{\lambda_j^2},$$

$$\Psi_{13} = \frac{\mu}{\alpha_0(\lambda_1^2 - \lambda_2^2)} \sum_1^2 (-1)^j m_{j3}(\varphi_j - \ln r),$$

$$m_{j3} = \beta_1 k_2 - \beta_2 k_{21} - \frac{\beta_1 a_2 - \beta_2 a_{21}}{\lambda_j^2}, \quad j = 1, 2,$$

$$\Psi_{31} = \frac{i\omega\mu}{\alpha_0(\lambda_1^2 - \lambda_2^2)} \sum_1^2 (-1)^j m_{3j}(\varphi_j - \ln r),$$

$$m_{3j} = \beta_1 k_2 - \beta_2 k_{12} - \frac{\beta_1 a_2 - \beta_2 a_{12}}{\lambda_j^2}, \quad j = 1, 2,$$

$$\Psi_{14} = \frac{\mu}{\alpha_0(\lambda_1^2 - \lambda_2^2)} \sum_1^2 (-1)^j m_{j4}(\varphi_j - \ln r),$$

$$m_{j4} = \beta_1 k_{12} - \beta_2 k_1 - \frac{\beta_1 a_{12} - \beta_2 a_1}{\lambda_j^2}, \quad j = 1, 2,$$

$$\Psi_{41} = \frac{i\omega\mu}{\alpha_0(\lambda_1^2 - \lambda_2^2)} \sum_1^2 (-1)^j m_{4j}(\varphi_j - \ln r)$$

$$m_{4j} = \beta_1 k_{21} - \beta_2 k_1 - \frac{\beta_1 a_{21} - \beta_2 a_1}{\lambda_j^2}, \quad j = 1, 2.$$

Clearly

$$\frac{\pi}{2i} H_0^{(1)}(\lambda r) = \ln|\mathbf{x} - \mathbf{y}| - \frac{\lambda^2}{4}|\mathbf{x} - \mathbf{y}|^2 \ln|\mathbf{x} - \mathbf{y}| + const + O(|\mathbf{x} - \mathbf{y}|^2).$$

It is evident that all elements of $\mathbf{\Gamma}(\mathbf{x}\text{-}\mathbf{y})$ are single-valued functions on the whole plane and they have a logarithmic singularity at most. It can be shown that columns of the matrix $\mathbf{\Gamma}(\mathbf{x}\text{-}\mathbf{y})$ are solutions to the system (3) with respect to $\mathbf{x}$ for any $\mathbf{x} \neq \mathbf{y}$. By applying the methods, as in the classical theory of elasticity, we can directly prove the following;

Theorem 3. *The elements of the matrix $\mathbf{\Gamma}(\boldsymbol{x}\text{-}\boldsymbol{y})$ have a logarithmic singularity as $\boldsymbol{x} \to \boldsymbol{y}$ and each column of the matrix $\mathbf{\Gamma}(\boldsymbol{x}\text{-}\boldsymbol{y})$,considered as a vector, is a solution of the system (4) at every point $\boldsymbol{x}$ if $\boldsymbol{x} \neq \boldsymbol{y}$.*

Let us consider the matrix $\widetilde{\mathbf{\Gamma}}(\mathbf{x}) := \mathbf{\Gamma}^T(-\mathbf{x})$. The following basic properties of $\widetilde{\mathbf{\Gamma}}(\mathbf{x})$ may be easily verified:

Theorem 4. *Each column of the matrix $\widetilde{\mathbf{\Gamma}}(\boldsymbol{x}\text{-}\boldsymbol{y})$, considered as a vector, satisfies the associated system $\widetilde{\boldsymbol{A}}(\partial\boldsymbol{x})\widetilde{\mathbf{\Gamma}}(\boldsymbol{x}\text{-}\boldsymbol{y}) = 0$, at every point $\boldsymbol{x}$ if $\boldsymbol{x} \neq \boldsymbol{y}$ and the elements of the matrix $\widetilde{\mathbf{\Gamma}}(\boldsymbol{x}\text{-}\boldsymbol{y})$ have a logarithmic singularity as $\boldsymbol{x} \to \boldsymbol{y}$.*

4. Singular matrix of solutions

Using the basic fundamental matrix, we will construct the so-called singular matrix of solutions and study their properties.

Write now the expressions for the components of the stress vector, which acts on an elements of the arc with the normal $\mathbf{n} = (n_1, n_2)$. Denoting the stress vector by $\boldsymbol{P}(\partial\mathbf{x}, \mathbf{n})\mathbf{u}$, we have

$$\boldsymbol{P}(\partial\mathbf{x}, \mathbf{n})\mathbf{u} = \boldsymbol{T}(\partial\mathbf{x}, \mathbf{n})\mathbf{u} - \mathbf{n}(\beta_1 p_1 + \beta_2 p_2), \tag{6}$$

where

$$\boldsymbol{T}(\partial\mathbf{x}, \mathbf{n})\mathbf{u} = \begin{pmatrix} \mu\dfrac{\partial}{\partial n} + (\lambda+\mu)n_1\dfrac{\partial}{\partial x_1} & (\lambda+\mu)n_1\dfrac{\partial}{\partial x_2} + \mu\dfrac{\partial}{\partial s} \\ (\lambda+\mu)n_2\dfrac{\partial}{\partial x_1} - \mu\dfrac{\partial}{\partial s} & \mu\dfrac{\partial}{\partial n} + (\lambda+\mu)n_2\dfrac{\partial}{\partial x_2} \end{pmatrix}\mathbf{u},$$

$$\frac{\partial}{\partial s} = n_2\frac{\partial}{\partial x_1} - n_1\frac{\partial}{\partial x_2}.$$

We introduce the following notation $\boldsymbol{R}(\partial\mathbf{x}, \mathbf{n})$, $\widetilde{\boldsymbol{R}}(\partial\mathbf{x}, \mathbf{n})$

$$\boldsymbol{R}(\partial\mathbf{x}, \mathbf{n}) = \begin{pmatrix} T_{11}(\partial x, n) & T_{12}(\partial x, n) & -\beta_1 n_1 & -\beta_2 n_1 \\ T_{21}(\partial x, n) & T_{22}(\partial x, n) & -\beta_1 n_2 & -\beta_2 n_2 \\ 0 & 0 & k_1\dfrac{\partial}{\partial n} & k_{12}\dfrac{\partial}{\partial n} \\ 0 & 0 & k_{21}\dfrac{\partial}{\partial n} & k_2\dfrac{\partial}{\partial n} \end{pmatrix},$$

$$\widetilde{\boldsymbol{R}}(\partial\mathbf{x}, \mathbf{n}) = \begin{pmatrix} T_{11}(\partial x, n) & T_{12}(\partial x, n) & -i\omega n_1\beta_1 & -i\omega n_1\beta_2 \\ T_{21}(\partial x, n) & T_{22}(\partial x, n) & -i\omega n_2\beta_1 & -i\omega n_2\beta_2 \\ 0 & 0 & k_1\dfrac{\partial}{\partial n} & k_{21}\dfrac{\partial}{\partial n} \\ 0 & 0 & k_{12}\dfrac{\partial}{\partial n} & k_2\dfrac{\partial}{\partial n} \end{pmatrix},$$

By Applying the operator $\boldsymbol{R}(\partial\mathbf{x},\mathbf{n})$ to the matrix $\boldsymbol{\Gamma}(\mathbf{x}\text{-}\mathbf{y}$ and the operator $\widetilde{\boldsymbol{R}}(\partial\mathbf{x},\mathbf{n})$ to the matrix $\widetilde{\boldsymbol{\Gamma}}(\mathbf{x}\text{-}\mathbf{y})$, we shall construct the so-called singular matrix of solutions respectively

$$\boldsymbol{R}(\partial\mathbf{x},\mathbf{n})\boldsymbol{\Gamma}(\mathbf{x}-\mathbf{y}) = \|R_{pq}\|_{4\times4}, \quad \widetilde{R}(\partial x,n)\widetilde{\Gamma}(y-x) = \|\widetilde{R}_{pq}\|_{4\times4},$$

The elements R_{pq} are following:

$$\begin{aligned}
&R_{pp} = \frac{\partial \ln r}{\partial n} + (-1)^p\, 2\mu\frac{\partial^2}{\partial x_1\partial x_2}\frac{\partial\Psi_{11}}{\partial s}, \quad p=1,2,\\
&R_{12} = \frac{\partial \ln r}{\partial s} - 2\mu\frac{\partial^2}{\partial x_2^2}\frac{\partial\Psi_{11}}{\partial s}, \quad R_{21} = -\frac{\partial \ln r}{\partial s} + 2\mu\frac{\partial^2}{\partial x_1^2}\frac{\partial\Psi_{11}}{\partial s},\\
&R_{13} = 2\mu\frac{\partial}{\partial x_2}\frac{\partial\Psi_{13}}{\partial s}, \quad R_{23} = -2\mu\frac{\partial}{\partial x_1}\frac{\partial\Psi_{13}}{\partial s}, \quad R_{14} = -2\mu\frac{\partial}{\partial x_2}\frac{\partial\Psi_{14}}{\partial s},\\
&R_{24} = 2\mu\frac{\partial}{\partial x_1}\frac{\partial\Psi_{14}}{\partial s}, \qquad R_{3j} = \frac{\partial}{\partial x_j}\frac{\partial}{\partial n}(k_{12}\Psi_{41} - k_1\Psi_{31}),\\
&R_{4j} = \frac{\partial}{\partial x_j}\frac{\partial}{\partial n}(k_2\Psi_{41} - k_{21}\Psi_{31}), \quad j=1,2,\\
&R_{33} = \frac{\mu}{\alpha_0(\lambda_1^2-\lambda_2^2)}\frac{\partial}{\partial n}\left\{(k_1m_{12} - k_{12}n_{22})\varphi_2 - (k_1m_{11} - k_{12}n_{11})\varphi_1\right\},\\
&R_{44} = \frac{\mu}{\alpha_0(\lambda_1^2-\lambda_2^2)}\frac{\partial}{\partial n}\left\{(k_2m_{22} - k_{21}n_{12})\varphi_2 - (k_2m_{21} - k_{21}n_{11})\varphi_1\right\},\\
&R_{34} = \frac{\mu}{\alpha_0(\lambda_1^2-\lambda_2^2)}(m_{22}k_{12} - k_1n_{12})\frac{\partial(\varphi_2-\varphi_1)}{\partial n},\\
&R_{43} = \frac{\mu}{\alpha_0(\lambda_1^2-\lambda_2^2)}(m_{12}k_{21} - k_2n_{22})\frac{\partial(\varphi_2-\varphi_1)}{\partial n},
\end{aligned}$$

Similarly we obtain the matrix

$$\widetilde{R}(\partial x,n)\widetilde{\Gamma}(y-x) = \|\widetilde{R}_{pq}\|_{4\times4},$$

where

$$\begin{aligned}
&\widetilde{R}_{pq} = R_{pq}, \quad p,q=1,2, \quad \widetilde{R}_{13} = 2\mu\frac{\partial}{\partial x_2}\frac{\partial\psi_{31}}{\partial s} \quad \widetilde{R}_{14} = -2\mu\frac{\partial}{\partial x_2}\frac{\partial\psi_{41}}{\partial s},\\
&\widetilde{R}_{23} = -2\mu\frac{\partial}{\partial x_1}\frac{\partial\psi_{31}}{\partial s}, \quad \widetilde{R}_{24} = 2\mu\frac{\partial}{\partial x_1}\frac{\partial\psi_{41}}{\partial s}, \quad \widetilde{R}_{3j} = \frac{\partial}{\partial n}\frac{\partial(k_{21}\psi_{14} - k_1\psi_{13})}{\partial x_j},\\
&\widetilde{R}_{4j} = \frac{\partial}{\partial n}\frac{\partial(-k_{12}\psi_{13} + k_2\psi_{14})}{\partial x_j}, \quad j=1,2,\\
&\widetilde{R}_{34} = \frac{\mu}{\alpha_0(\lambda_1^2-\lambda_2^2)}(k_{21}m_{22} - k_1n_{22})\frac{\partial}{\partial n}(\varphi_2-\varphi_1),\\
&\widetilde{R}_{43} = \frac{\mu}{\alpha_0(\lambda_1^2-\lambda_2^2)}(k_{12}m_{12} - k_2n_{12})\frac{\partial}{\partial n}(\varphi_2-\varphi_1),\\
&\widetilde{R}_{33} = \frac{\mu}{\alpha_0(\lambda_1^2-\lambda_2^2)}\frac{\partial}{\partial n}\left\{(k_1m_{12} - k_{21}n_{12})\varphi_2 - (k_1m_{11} - k_{21}n_{11})\varphi_1\right\},\\
&\widetilde{R}_{44} = \frac{\mu}{\alpha_0(\lambda_1^2-\lambda_2^2)}\frac{\partial}{\partial n}\left\{(k_2m_{22} - k_{21}n_{22})\varphi_2 - (k_2m_{21} - k_{21}n_{21})\varphi_1\right\},
\end{aligned}$$

Let us consider the matrix $[\boldsymbol{R}(\partial\mathbf{y},\mathbf{n})\boldsymbol{\Gamma}(\mathbf{y}-\mathbf{x})]^{*}$, which is obtained from $\boldsymbol{R}(\partial\mathbf{x},\mathbf{n})\boldsymbol{\Gamma}(\mathbf{x}-\mathbf{y})=(R_{pq})_{4\times 4}$ by transposition of the columns and rows and the variables x and y (analogously $\left[\tilde{\boldsymbol{R}}(\partial\mathbf{y},\mathbf{n})\tilde{\boldsymbol{\Gamma}}(\mathbf{y}\text{-}\mathbf{x})\right]^{T}$.) We can state the following:.

Theorem 5. *Every column of the matrix* $[\boldsymbol{R}(\partial\boldsymbol{y},\boldsymbol{n})\boldsymbol{\Gamma}(\boldsymbol{y}\text{-}\boldsymbol{x})]^{T}$ *, considered as a vector, is a solution of the system* $\widetilde{\boldsymbol{A}}(\partial\boldsymbol{x})=0$ *at any point* $\boldsymbol{x}$ *if* $\boldsymbol{x}\neq\boldsymbol{y}$ *and the elements of the matrix*$[\mathbf{R}(\partial\boldsymbol{y},\boldsymbol{n})\boldsymbol{\Gamma}(\boldsymbol{y}\text{-}\boldsymbol{x})]^{T}$ *contain a singular part, which is integrable in the sense of the Cauchy principal value.*

Theorem 6. *Every column of the matrix* $\left[\tilde{\boldsymbol{R}}(\partial\boldsymbol{y},\boldsymbol{n})\tilde{\boldsymbol{\Gamma}}(\boldsymbol{y}\text{-}\boldsymbol{x})\right]^{T}$ *, considered as a vector, is a solution of the system* $\boldsymbol{A}(\partial\boldsymbol{x})\boldsymbol{U}=0$ *at any point* $\boldsymbol{x}$ *if* $\boldsymbol{x}\neq\boldsymbol{y}$ *and the elements of the matrix* $\left[\tilde{\boldsymbol{R}}(\partial\boldsymbol{y},\boldsymbol{n})\tilde{\boldsymbol{\Gamma}}(\boldsymbol{y}\text{-}\boldsymbol{x})\right]^{T}$*, contain a singular part, which is integrable in the sense of the Cauchy principal value.*

Let us introduce the following single and double layer potentials : The vector-functions defined by the equalities

$$\mathbf{V}(\mathbf{x};\mathbf{g})=\frac{1}{\pi}\int\limits_{S}\boldsymbol{\Gamma}(\mathbf{x}-\mathbf{y})\mathbf{g}(\mathbf{y})d_{\mathbf{y}}S,$$

$$\widetilde{\boldsymbol{V}}(\mathbf{x};\mathbf{g})=\frac{1}{\pi}\int\limits_{S}\boldsymbol{\Gamma}^{T}(\mathbf{y}-\mathbf{x})\mathbf{g}(\mathbf{y})d_{\mathbf{y}}S$$

will be called single layer potentials, while the vector-functions defined by the equalities

$$\mathbf{W}(\mathbf{x};\mathbf{h})=\frac{1}{\pi}\int\limits_{S}\left[\boldsymbol{P}(\partial_{\mathbf{y}},\mathbf{n})\boldsymbol{\Gamma}(\mathbf{y}-\mathbf{x})\right]^{T}\mathbf{h}(\mathbf{y})d_{\mathbf{y}}S,$$

$$\widetilde{\boldsymbol{W}}(\mathbf{x};\mathbf{h})=\frac{1}{\pi}\int\limits_{S}\left[\widetilde{\boldsymbol{P}}(\partial_{\mathbf{z}},\mathbf{n})\boldsymbol{\Gamma}^{T}(\mathbf{x}-\mathbf{y})\right]^{T}\mathbf{h}(\mathbf{y})d_{\mathbf{y}}S$$

will be called double layer potentials. Here $\mathbf{g}$ and $\mathbf{h}$ are the continuous (or Hölder continuous) vectors and S is a closed Lyapunov curve.

We can state the following:

Theorem 7. *The vector* $\boldsymbol{W}(\boldsymbol{x};\mathbf{h})$ *is a solution of the system* $\widetilde{\boldsymbol{A}}(\partial_{\boldsymbol{x}})\boldsymbol{U}=\mathbf{0}$ *at any point* $\boldsymbol{x}$ *and* $\boldsymbol{x}\neq\boldsymbol{y}$*. The elements of the matrix* $[\boldsymbol{P}(\partial_{\boldsymbol{y}},\boldsymbol{n})\boldsymbol{\Gamma}(\boldsymbol{y}-\mathbf{x})]^{T}$ *contain a singular part, which is integrable in the sense of the Cauchy principal value.*

Theorem 8. *The vector* $\widetilde{\boldsymbol{W}}(\boldsymbol{x};\mathbf{h})$ *is a solution of the system* $\boldsymbol{A}(\partial_{\boldsymbol{x}})\boldsymbol{U}=\mathbf{0}$ *at any point* $\boldsymbol{x}$ *and* $\boldsymbol{x}\neq\boldsymbol{y}$*. The elements of the matrix* $\left[\widetilde{\boldsymbol{P}}(\partial_{\boldsymbol{y}},\boldsymbol{n})\boldsymbol{\Gamma}^{T}(\boldsymbol{x}-\mathbf{y})\right]^{T}$ *contain a singular part, which is integrable in the sense of the Cauchy principal value.*

Theorem 9. *If* $S\in C^{1,\eta}(S)$, $\boldsymbol{g},\mathbf{h}\in C^{0,\delta}(S)$, $\quad 0<\delta<\eta\leq 1$, *then the vectors* $\boldsymbol{W}(\boldsymbol{x},\boldsymbol{h})$, $\boldsymbol{V}(\boldsymbol{x},\boldsymbol{g})$, $\widetilde{\boldsymbol{W}}(\boldsymbol{x},\boldsymbol{h})$ *and* $\widetilde{\boldsymbol{V}}(\boldsymbol{x},\boldsymbol{g})$ *are the regular vector-functions in* $D^{+}(D^{-})$*, and when the point* $\boldsymbol{x}$ *tends to any point* $\boldsymbol{z}$ *of the boundary* S *from inside or from* *outside we have the following formulas:*

$$[\mathbf{W}(\mathbf{z},\mathbf{h})]^{\pm} = \mp\mathbf{h}(\mathbf{z}) + \frac{1}{\pi}\int\limits_S \left[\boldsymbol{P}(\partial_{\mathbf{y}},\mathbf{n})\boldsymbol{\Gamma}(\mathbf{y}-\mathbf{z})\right]^T \mathbf{h}(\mathbf{y})d_{\mathbf{y}}S,$$

$$[\widetilde{\boldsymbol{W}}(\mathbf{z},\mathbf{h})]^{\pm} = \mp\mathbf{h}(\mathbf{z}) + \frac{1}{\pi}\int\limits_S \left[\widetilde{\boldsymbol{P}}(\partial_{\mathbf{y}},\mathbf{n})\boldsymbol{\Gamma}^T(\mathbf{z}-\mathbf{y})\right]^T \mathbf{h}(\mathbf{y})d_{\mathbf{y}}S,$$

$$[\boldsymbol{P}(\partial_{\mathbf{z}},\mathbf{n})\mathbf{V}(\mathbf{z},\mathbf{g})]^{\pm} = \pm\mathbf{g}(\mathbf{z}) + \frac{1}{\pi}\int\limits_S \boldsymbol{P}(\partial_{\mathbf{z}},\mathbf{n})\boldsymbol{\Gamma}(\mathbf{z}-\mathbf{y})\mathbf{g}(\mathbf{y})d_{\mathbf{y}}S,$$

$$[\widetilde{\boldsymbol{P}}(\partial_{\mathbf{z}},\mathbf{n})\widetilde{\boldsymbol{V}}(\mathbf{z},\mathbf{g})]^{\pm} = \pm\mathbf{g}(\mathbf{z}) + \frac{1}{\pi}\int\limits_S \widetilde{\boldsymbol{P}}(\partial_{\mathbf{z}},\mathbf{n})\boldsymbol{\Gamma}^T(\mathbf{y}-\mathbf{z})\mathbf{g}(\mathbf{y})d_{\mathbf{y}}S.$$

Here the integrals are singular and understood as the principal value.

Theorem 10. *The potentials* $\boldsymbol{V}(\boldsymbol{x},\boldsymbol{g})$ *and* $\widetilde{\boldsymbol{W}}(\boldsymbol{x},\boldsymbol{h})$ *are solutions of the system* $\boldsymbol{A}(\partial_{\boldsymbol{x}})\boldsymbol{U} = \mathbf{0}$ *and the potentials* $\widetilde{\boldsymbol{V}}(\boldsymbol{x},\boldsymbol{g})$ *and* $\boldsymbol{W}(\boldsymbol{x},\boldsymbol{h})$ *are solutions of the system* $\widetilde{\boldsymbol{A}}(\partial_{\boldsymbol{x}})\boldsymbol{U} = \mathbf{0}$ *in both domains* D^+ *and* D^-*.*

R E F E R E N C E S

1. Wilson D R.K., Aifantis E.C. On the theory of Consolidation with Double Porosity-I. *Int. J. Engng. Sci.*, **20** (1982), 1009-1035.

2. Beskos D.E., Aifantis E.C. On the theory of consolidation with double porosity-II. *Int. J. Engng. Sci.*, **24** (1986), 1697-1716.

3. khaled M.Y., Beskos D.E., Aifantis E.C. On the theory of consolidation with double porosity-III, A finite element formulation. *Int. J. for Numerical and Analytical Methods in Geomechanics*, **8**, 2 (1984), 101-123.

4. Barenblatt G.I., Zheltov I.P., Kochina I.N. basic concept in the theory of seepage of homogeneous liquids in fissured rocks.*j. of Appl. Math. and Mech.*, **24** (1960), 1286-1303.

5. Biot M.A. General theory of three-dimensional consolidation. *J. of Appl. Physics*, **12** (1941), 155-164.

6. Khalili N., Valliappan S. Unified theory of flow and deformation on double porous media. *Europian J. of Mechanics,A/Solids*, **15** (1996), 321-336.

7. Pride S.R., Berryman J.G. Linear dynamics of double porosity dual-permeability materials. *I, physical Review. E*, **68** (2003), 036603.

8. Bai M., Elsworth D., Roegiers J.C. Multiporosity/multipermeability approach to the simulation of naturally fractured reservoirs. *Water Resources Research*, **29** (1993), 1621-1633.

9. Moutsopoulos K.N., Konstantinidis A.A., Meladiotis I., Tzimopoulos Ch.D., Aifantis E.C. Hydraulic behavior and contaminant transport in multiple porosity media. *Transport Porous Media*, **42** (2001), 265-292.

10. Bai M., J.C. Roegiers J.C. Fluid flow and heat flow in deformable fractured porous media. *International Journal of Engineering Science*, **32** (1994), 1615-1633.

11. Khalili N., Selvadurai A.P.S. A fully coupled constitutive model for thermo-hydromechanical analysis in elastic media with double porosity. *Geophysical Research Letters*, **30** (2003), 22-68.

12. Masters I., Pao W.K.S., Lewis R.W. Coupling temperature to a double-porosity model of deformable porous media. *International Journal for Numerical Methods in Engineering*, **49** (2000), 421438.

13. Cowin S.C., Bone poroelasticity. *Journal of Biomechanics*, **32** (1999), 217-238.

14. Cowin S.C., Gailani G., Benalla M. Hierarchical poroelasticity: movement of interstitial fluid between levels in bone. *Philosophical Transactions of the Royal Society A: mathematical, physical and engineering sciences*, **367**, (2009), 34013444.

15. Rohan E., Naili S., Cimrman R., Lemaire T. Multiscale modeling of a fluid saturated medium with double porosity: Relevance to the compact bone. *Journal of the Mechanics and Physics of Solids*, **60** (2012), 857-881.

16. De Boer R. Theory of Porous Media: Highlights in the historical development and current state, Springer, Berlin, Heidelberg, New York, 2000.

17. De Boer R., Svanadze M. Fundamental solution of the system of equations of steady oscillations in the theory of fluid-saturated porous media. T*ransport in Porous Media*, **56**, (2004), 3950.

18. Svanadze M. Fundamental solution in the theory of consolidation with double porosity. *Journal of the Mechanical Behavior of Materials*, **16**, (2005), 123-130.

19. Basheleishvili M., Bitsadze L. Two dimensional BVPs of the theory of consolidation with double porosity. *Mem. Differential Equations Math. Phys.*, **51** (2010), 43-58.

20. Bitsadze L. Fundamental Solution in the Theory of Poroelasticity of Steady Vibrations for Solids with Double Porosity. *Proc.of I. Vekua Inst. of Appl. Math.*, **64** (2014), 3-12.

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