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BOUNDARY VALUE PROBLEMS OF THE FULLY COUPLED THEORY OF ELASTICITY FOR SOLIDS WITH DOUBLE POROSITY FOR HALF-PLANE

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Abstract. In the paper the two-dimensional version of steady vibration in the fully coupled linear theory of elasticity for solids with double porosity is considered. Using the Fourier integrals, some basic boundary value problems are solved explicitly (in quadratures) for the half-plane.

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Introduction

Porous media theories play an important role in many branches of engineering, including material science, the petroleum industry, chemical engineering, and soil mechanics, as well as biomechanics.

In a material with two degrees of porosity, there are two pore systems, the primary and the secondary. For example in a fissured rock (i.e., a mass of porous blocks separated from each other by an interconnected and continuously distributed system of fissures) most of the porosity is provided by the pores of the blocks or primary porosity, while most of permeability is provided by the fissures or secondary porosity. When fluid flow and deformations processes occur simultaneously, three coupled partial differential equations can be derived [1],[2] to describe the relationships governing pressure in the primary and secondary pores (and therefore the mass exchange between them) and the displacement of the solid.

A theory of consolidation with double porosity has been proposed by Aifantis [1]. The physical and mathematical foundations of the theory of double porosity were considered in the papers [1],[2], [3], where analytical solutions of the relevant equations are also given.This theory unifies a model proposed by Biot for the consolidation of deformable single porosity media with a model proposed by Barenblatt for seepage in undeformable media with two degrees of porosity. The basic results and the historical information on the theory of porous media were summarized by R.de Boer [4]. However, Aifantis' quasi-static theory ignored the cross-coupling effect between the volume change of the pores and fissures in the system. The cross-coupled terms were included in the equations of conservation of mass for the pore and fissure fluid and in Darcy's law for solids with double porosity by several authors [5-8].

In the last years many authors have investigated different types of problems of the 2-dimensional and 3-dimensional theories of elasticity for materials with double porosity, publishing a large number of papers (some of these results can be seen in [9-20] and references therein). There the explicit solutions on some BVPs in the form of series and in quadratures are given in a form useful for engineering practice.

The purpose of this paper is to consider the two-dimensional version of steady vibration in the fully coupled linear theory of elasticity for solids with double porosity. Using the Fourier integrals, some basic boundary value problems in the fully coupled linear theory of elasticity are solved explicitly (in quadratures) for the half-plane.

2. Basic equations. Boundary value problems

Let R^2_+ denote the upper half-plane $x_2 > 0$. The boundary of R^2_+ which is x_1-axis we denoted by S : Let $\mathbf{x} := (x_1, x_2) \in R^2_+$, $\quad \partial\mathbf{x} := \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)$. We assume the domain R^2_+ to be filled with an isotropic elastic material with double porosity.

The governing homogeneous system of the theory of steady vibration in the fully coupled linear theory of elasticity for materials with double porosity has the form [9]

$$\begin{aligned}
&\mu\Delta\mathbf{u} + (\lambda+\mu) graddiv\mathbf{u} - grad(\beta_1 p_1 + \beta_2 p_2) + \rho_1\omega^2\mathbf{u} = 0,\\
&i\omega\beta_1 div\mathbf{u} + (k_1\Delta + a_1)p_1 + (k_{12}\Delta + a_{12})p_2 = 0, \qquad (1)\\
&i\omega\beta_2 div\mathbf{u} + (k_{21}\Delta + a_{21})p_1 + (k_2\Delta + a_2)p_2 = 0,
\end{aligned}$$

where $\mathbf{u}(\mathbf{x}) = \mathbf{u}(u_1, u_2)$ is the displacement vector in a solid, $p_1(\mathbf{x})$ and $p_2(\mathbf{x})$ are the pore and fissure fluid pressures respectively. $a_j = i\omega\alpha_j - \gamma$, $\quad \omega > 0 \quad$ is the oscillation frequency, $\quad \rho_1 > 0 \quad$ is the reference mass density, β_1 and β_2 are the effective stress parameters, $\gamma > 0$ is the internal transport coefficient and corresponds to fluid transfer rate with respect to the intensity of flow between the pore and fissures, λ, μ, are constitutive coefficients, α_1 and α_2 measure the compressibilities of the pore and fissure system, respectively. $k_j = \frac{\kappa_j}{\mu'}$, $k_{12} = \frac{\kappa_{12}}{\mu'}$, $k_{21} = \frac{\kappa_{21}}{\mu'}$. μ' is the fluid viscosity, κ_1 and κ_2 are the macroscopic intrinsic permeabilities associated with matrix and fissure porosity, respectively, κ_{12} and κ_{21} are the cross-coupling permeabilities for fluid flow at the interface between the matrix and fissure phases, Δ is the Laplace operator. Throughout this article it is assumed that $\beta_1^2 + \beta_2^2 > 0$. Vectors, if needed, we consider as column matrices.

Here we state the following BVPs.

Find a solution $\boldsymbol{U}(\boldsymbol{u}, p_1, p_2) \in C^2(R^2_+)$ to the Eqs. (1) in R^2_+ , satisfying one of the following boundary conditions (BCs) on S :

Problem 1.

$$\mathbf{u}^+ = \mathbf{f}(x_1), \quad p_1^+ = f_3(x_1), \quad p_2^+ = f_4(x_1), \quad x_1 \in S, \qquad (2)$$

Problem 2.

$$\begin{aligned}
&u_1^+ = f_1(x_1),\\
&(\mathbf{P}(\partial\mathbf{x},\mathbf{n})\mathbf{u})_2^+ = f_2(x_1), \quad p_1 = f_3(x_1), \quad p_2 = f_4(x_1), \qquad (3)
\end{aligned}$$

Problem 3.

$$(\mathbf{P}(\partial\mathbf{x},\mathbf{n})\mathbf{u})_1^+ = f_1(x_1), \quad u_2^+ = f_2(x_1), \quad \frac{\partial p_1}{\partial x_2} = f_3(x_1), \quad \frac{\partial p_2}{\partial x_2} = f_4(x_1). \qquad (4)$$

The symbol $(.)^+$ denotes the limit on S from R^2_+,

$$\lim_{R^2_+\ni\mathbf{x}\to x_1\in S}\mathbf{u}=\mathbf{f}(x_1),\qquad \lim_{R^2_+\ni\mathbf{x}\to x_1\in S}p_1=\mathbf{f}_3(x_1),\qquad \lim_{R^2_+\ni\mathbf{x}\to x_1\in S}p_2=\mathbf{f}_4(x_1),$$
$$\lim_{R^2_+\ni\mathbf{x}\to x_1\in S}\left[\mathbf{P}\left(\frac{\partial}{\partial\mathbf{x}},\mathbf{n}\right)\mathbf{U}\right]_\alpha=f_\alpha(x_1),\quad \alpha=1,2,$$

the functions f_j, $j=1,2,3,4$, are prescribed, $\mathbf{n}:=(0,1)$ is a unit normal vector,

$$\mathbf{P}(\partial\mathbf{x},\mathbf{n})\mathbf{U}=\mathbf{T}(\partial\mathbf{x},\mathbf{n})\mathbf{u}-\mathbf{n}(\beta_1p_1+\beta_2p_2), \tag{5}$$

$\mathbf{T}(\partial\mathbf{x},\mathbf{n})\mathbf{u}$ is the following vector

$$\mathbf{T}(\partial\mathbf{x},\mathbf{n})\mathbf{u}:=\begin{pmatrix}\mu\frac{\partial}{\partial x_2} & \mu\frac{\partial}{\partial x_1}\\ \lambda\frac{\partial}{\partial x_1} & \mu_0\frac{\partial}{\partial x_2}\end{pmatrix}\mathbf{u},\quad \mu_0:=\lambda+2\mu.$$

In the domain of regularity the regular solution $\mathbf{U}=(\mathbf{u},p_1,p_2)\in C^2(D)$ of system (1) is represented as the sum (see appendix 1)

$$\begin{aligned}&\mathbf{u}(\mathbf{x})=-grad\sum_{m=1}^{3}\frac{\varphi_m(\mathbf{x})}{\lambda_m^2}+\mathbf{u}^{(4)}(\mathbf{x}),\quad div\mathbf{u}^{(4)}(\mathbf{x})=0,\\ &p_1(\mathbf{x})=\sum_{m=1}^{3}B_m\varphi_m(\mathbf{x}),\quad p_2(\mathbf{x})=\sum_{m=1}^{3}C_m\varphi_m(\mathbf{x}),\end{aligned} \tag{6}$$

where

$$(\Delta+\lambda_m^2)\varphi_m(\mathbf{x})=0,\quad (\Delta+\lambda_4^2)\mathbf{u}^{(4)}(\mathbf{x})=0,\quad div\mathbf{u}^{(4)}(\mathbf{x})=0,$$
$$B_m=-\frac{i\omega}{\delta_m}\left[\beta_1(a_2-k_2\lambda_m^2)-\beta_2(a_{12}-k_{12}\lambda_m^2)\right],$$
$$C_m=-\frac{i\omega}{\delta_m}\left[\beta_2(a_1-k_1\lambda_m^2)-\beta_1(a_{21}-k_{21}\lambda_m^2)\right],$$
$$\delta_m=(k_1k_2-k_{12}k_{21})\lambda_m^4-k_0\lambda_m^2+a_1a_2-a_{12}a_{21},$$
$$\beta_1B_m+\beta_2C_m=-\frac{i\omega}{\delta_m}(\alpha_{12}-\alpha_{11}\lambda_m^2).$$

λ_j^2, $j=1,2,3$, are roots of cubic algebraic equation

$$\begin{aligned}&\mu_0\alpha_0\xi^3-[\mu_0k_0+i\omega\alpha_{11}+\rho_1\omega^2\alpha_0]\xi^2\\ &+[\mu_0(a_1a_2-a_{12}a_{21})+i\omega\alpha_{12}+\rho_1\omega^2k_0]\xi-\rho_1\omega^2(a_1a_2-a_{12}a_{21})=0,\\ &\alpha_{11}=k_2\beta_1^2+k_1\beta_2^2-\beta_1\beta_2(k_{12}+k_{21}),\alpha_{12}=a_2\beta_1^2+a_1\beta_2^2-\beta_1\beta_2(a_{12}+a_{21}),\\ &\alpha_0=k_1k_2-k_{12}k_{21},k_0=a_1k_2+a_2k_1-k_{12}a_{21}-k_{21}a_{12},\lambda_4^2=\frac{\rho_1\omega^2}{\mu}.\end{aligned} \tag{7}$$

Let us assume that

$$\widehat{\mathbf{F}}(x_1)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\mathbf{F}(\xi)\exp(-ix_1\xi)d\xi$$

and the inversion formula

$$\mathbf{F}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \widehat{\mathbf{F}}(x_1) \exp(ix_1\xi) dx_1$$

is valid.

The Fourier integral theorem holds if both $\mathbf{F}$ and its Fourier transform are absolutely integrable and $\mathbf{F}$ is bounded and continuous at the point x_1. [24]

In what follows we assume, that the vector $\mathbf{f}$, and the functions f_3, f_4 are absolutely integrable, bounded, and continuous on S, moreover $\widehat{\mathbf{f}}$, $\widehat{f}_3$, and $\widehat{f}_4$ are absolutely integrable on S.

Theorem 1. *Problem 1 has at most one regular solution in the domain D.*

Theorem 1 can be proved similarly to the corresponding theorem in the classical theory of elasticity (for details see [25]).

Solution of Problem 1 for a half-plane

The solution of Problem 1 is sought in the form (6). Let us assume that the functions $\varphi_m(\mathbf{x})$, $m = 1, 2, 3$, and $\mathbf{u}^{(4)}(\mathbf{x})$ are sought in the form [23]

$$\begin{aligned} &\varphi_m = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \alpha_m(\xi) exp(-x_2 r_m) exp[ix_1\xi] d\xi, \quad k = 1, 2, 3, \\ &\mathbf{u}^{(4)}(\mathbf{x}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \boldsymbol{\alpha}^{(4)}(\xi) exp(-x_2 r_4) exp[ix_1\xi] d\xi, \\ &r_m^2 = \xi^2 - \lambda_m^2, \quad \boldsymbol{\alpha}^{(4)} = (\alpha_1^{(4)}, \alpha_2^{(4)}), \end{aligned} \tag{8}$$

where $\boldsymbol{\alpha}^{(4)}$ and α_m are absolutely integrable on S unknown values.

It is not difficult to prove that (8) satisfy equations $(\Delta + \lambda_m^2)\varphi_m = 0$, $m = 1, 2, 3$, $(\Delta + \lambda_4^2)\mathbf{u}^{(4)} = 0$ for arbitrary α_m and $\boldsymbol{\alpha}^{(4)}$, respectively.

By substituting in (6) the expressions of $\varphi_m(\mathbf{x})$ and $\mathbf{u}^{(4)}$ from (8), passing to the limit as $x_2 \to 0$, and taking into account boundary conditions, for determining the unknown values α_m, $k = 1, 2, 3$ and $\boldsymbol{\alpha}^{(4)}$, we obtain the following system of algebraic equations

$$\begin{aligned} &\xi^2 \sum_{m=1}^{3} \frac{\alpha_m}{\lambda_m^2} + r_4 \alpha_2^{(4)} = i\xi \widehat{f}_1, \qquad \sum_{m=1}^{3} \frac{r_m \alpha_m}{\lambda_m^2} + \alpha_2^{(4)} = \widehat{f}_2, \\ &\sum_{m=1}^{3} B_m \alpha_m = \widehat{f}_3, \qquad \sum_{m=1}^{3} C_m \alpha_m = \widehat{f}_4, \qquad i\xi\alpha_1^{(4)} - r_4 \alpha_2^{(4)} = 0. \end{aligned} \tag{9}$$

It easy to show that the determinant of system (9) has the form

$$\Delta_1 =$$

$$= -\omega^2 d \left\{ \frac{(\lambda_2^2 - \lambda_3^2)(r_4\lambda_1^2 + r_1\lambda_4^2)}{\lambda_1^2 \delta_2 \delta_3 (r_1 + r_4)} - \frac{(\lambda_1^2 - \lambda_3^2)(r_4\lambda_2^2 + r_2\lambda_4^2)}{\lambda_2^2 \delta_1 \delta_3 (r_2 + r_4)} + \frac{(\lambda_1^2 - \lambda_2^2)(r_4\lambda_3^2 + r_3\lambda_4^2)}{\lambda_3^2 \delta_1 \delta_2 (r_3 + r_4)} \right\}$$

$$d = (\beta_1 a_2 - \beta_2 a_{12})(\beta_2 k_1 - \beta_1 k_{21}) - (\beta_2 a_1 - \beta_1 a_{21})(\beta_1 k_2 - \beta_2 k_{12}).$$

Due to Theorem 1 we conclude that the determinant of system (9) different from zero and system (9) is uniquely solvable.

From (9) we find

$$\Delta_1\alpha_1 = -\left[i\xi\widehat{f}_1 - r_4\widehat{f}_2\right]\eta_1 + \left[\frac{C_3}{\lambda_2^2}(r_2r_4-\xi^2) - \frac{C_2}{\lambda_3^2}(r_3r_4-\xi^2)\right]\widehat{f}_3$$
$$-\left[\frac{B_3}{\lambda_2^2}(r_2r_4-\xi^2) - \frac{B_2}{\lambda_3^2}(r_3r_4-\xi^2)\right]\widehat{f}_4,$$
$$\Delta_1\alpha_2 = \left[i\xi\widehat{f}_1 - r_4\widehat{f}_2\right]\eta_2 - \left[\frac{C_3}{\lambda_1^2}(r_1r_4-\xi^2) - \frac{C_1}{\lambda_3^2}(r_3r_4-\xi^2)\right]\widehat{f}_3$$
$$+\left[\frac{B_3}{\lambda_1^2}(r_1r_4-\xi^2) - \frac{B_1}{\lambda_3^2}(r_3r_4-\xi^2)\right]\widehat{f}_4,$$
$$\Delta_1\alpha_3 = -\left[i\xi\widehat{f}_1 - r_4\widehat{f}_2\right]\eta_3 + \left[\frac{C_2}{\lambda_1^2}(r_1r_4-\xi^2) - \frac{C_1}{\lambda_2^2}(r_2r_4-\xi^2)\right]\widehat{f}_3$$
$$-\left[\frac{B_3}{\lambda_2^2}(r_2r_4-\xi^2) - \frac{B_2}{\lambda_3^2}(r_3r_4-\xi^2)\right]\widehat{f}_4,$$

$$\Delta_1\alpha_2^{(4)} = \left[\frac{r_1-r_3}{\lambda_1^2\lambda_3^2}B_2 + \frac{r_2-r_1}{\lambda_1^2\lambda_2^2}B_3 + \frac{r_3-r_2}{\lambda_2^2\lambda_3^2}B_1\right]\xi^2\widehat{f}_4$$
$$-\left[\frac{r_1-r_3}{\lambda_1^2\lambda_3^2}C_2 + \frac{r_2-r_1}{\lambda_1^2\lambda_2^2}C_3 + \frac{r_3-r_2}{\lambda_2^2\lambda_3^2}C_1\right]\xi^2\widehat{f}_3,$$
$$-\frac{\omega^2 d(a_1a_2-a_{12}a_{21})}{\lambda_1^2\lambda_2^2\lambda_3^2\delta_1\delta_2\delta_3}(\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_3^2)(\lambda_2^2-\lambda_3^2)\xi^2\widehat{f}_2,$$
$$+\left[\frac{r_1\eta_1}{\lambda_1^2} - \frac{r_2\eta_2}{\lambda_2^2} + \frac{r_3\eta_3}{\lambda_3^2}\right]i\xi\widehat{f}_1, \quad i\xi\alpha_1^{(4)} = r_4\alpha_2^{(4)},$$
$$\eta_1 = \frac{\omega^2 d}{\delta_2\delta_3}(\lambda_2^2-\lambda_3^2), \quad \eta_2 = \frac{\omega^2 d}{\delta_1\delta_3}(\lambda_1^2-\lambda_3^2), \quad \eta_3 = \frac{\omega^2 d}{\delta_1\delta_2}(\lambda_1^2-\lambda_2^2).$$

Substituting the obtained values in (6), we obtain the desired solution of the BVP in quadratures.

Solution of Problem 2 for a half-plane

A solution is sought in the form (6),(8). Keeping in mind BCs and

$$[Pu]_2 = -\frac{\mu}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left\{(r_4^2+\xi^2)\sum_{m=1}^{3}\frac{\alpha_m}{\lambda_m^2}exp(-x_2r_m) + 2r_4\alpha_2^{(4)}exp(-x_2r_4)\right\}exp(ix_1\xi)d\xi,$$

after passing to the limit, as $x_2 \to 0$, we get the following system of algebraic equations

$$\xi^2\sum_{m=1}^{3}\frac{\alpha_m}{\lambda_m^2} + r_4\alpha_2^{(4)} = i\xi\widehat{f}_1, \quad (r_4^2+\xi^2)\sum_{m=1}^{3}\frac{\alpha_m}{\lambda_m^2} + 2r_4\alpha_2^{(4)} = -\frac{\widehat{f}_2}{\mu},$$
$$\sum_{m=1}^{3}B_m\alpha_m = \widehat{f}_3, \quad \sum_{m=1}^{3}C_m\alpha_m = \widehat{f}_4, \quad i\xi\alpha_1^{(4)} - r_4\alpha_2^{(4)} = 0.$$

From here we obtain

$$\begin{aligned} &\sum_{m=1}^{3} \frac{\alpha_m}{\lambda_m^2} = \left[\frac{\widehat{f}_2}{\mu} + 2i\xi \widehat{f}_1\right] \frac{1}{\lambda_4^2}, \\ &\sum_{m=1}^{3} B_m \alpha_m = \widehat{f}_3, \quad \sum_{m=1}^{3} C_m \alpha_m = \widehat{f}_4, \end{aligned} \tag{10}$$

The determinant of system (10) has the form

$$D_2 = -\frac{\omega^2 d}{\lambda_1^2 \lambda_2^2 \lambda_3^2 \delta_1 \delta_2 \delta_3} [a_1 a_2 - a_{12} a_{21}](\lambda_1^2 - \lambda_2^2)(\lambda_1^2 - \lambda_3^2)(\lambda_2^2 - \lambda_3^2) \neq 0.$$

By elementary calculation, from (10) we obtain

$$\begin{aligned} &\alpha_m D_2 = (-1)^m \left\{ \left[\frac{\widehat{f}_2}{\mu} + 2i\xi \widehat{f}_1\right] \eta_m + c_m \widehat{f}_3 - b_m \widehat{f}_4 \right\}, \quad m = 1, 2, 3, \\ &\alpha_2^{(4)} = -\frac{1}{\lambda_4^2 r_4} \left[i\xi(\xi^2 + r_4^2)\widehat{f}_1 + \xi^2 \frac{\widehat{f}_2}{\mu} \right] \end{aligned}$$

where

$$\begin{aligned} &\eta_1 = \frac{\omega^2 d}{\lambda_4^2 \delta_2 \delta_3}(\lambda_2^2 - \lambda_3^2), \quad \eta_2 = \frac{\omega^2 d}{\lambda_4^2 \delta_1 \delta_3}(\lambda_1^2 - \lambda_3^2), \quad \eta_3 = \frac{\omega^2 d}{\lambda_4^2 \delta_1 \delta_2}(\lambda_1^2 - \lambda_2^2), \\ &c_1 = \frac{C_3}{\lambda_2^2} - \frac{C_2}{\lambda_3^2}, \quad c_2 = \frac{C_3}{\lambda_1^2} - \frac{C_1}{\lambda_3^2}, \quad c_3 = \frac{C_2}{\lambda_1^2} - \frac{C_1}{\lambda_2^2}, \\ &b_1 = \frac{B_3}{\lambda_2^2} - \frac{B_2}{\lambda_3^2}, \quad b_2 = \frac{B_3}{\lambda_1^2} - \frac{B_1}{\lambda_3^2}, \quad b_3 = \frac{B_2}{\lambda_1^2} - \frac{B_1}{\lambda_2^2}. \end{aligned}$$

Substituting the obtained values in (6) and taking into account the following formula [24]

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(-x_2 r_m) \exp[i\xi(x_1 - y_1)] \frac{1}{r_m} d\xi = i\sqrt{\frac{\pi}{2}} H_0^{(1)}(i\lambda_m r),$$

where $H_0^{(1)}(i\lambda_m r)$ is the first kind Hankel function of zero order,

$$r^2 = (x_1 - y_1)^2 + x_2^2, \quad r_m^2 = \xi^2 - \lambda_m^2, \quad m = 1, 2, 3$$

we obtain

$$\begin{aligned} \varphi_m = {} & \frac{i(-1)^{m+1}}{2D_2} \int_{-\infty}^{+\infty} \left[\frac{\eta_m}{\mu} f_2(y_1) + c_m f_3(y_1) - b_m f_4(y_1) \right] \frac{\partial}{\partial x_2} H_0^{(1)}(i\lambda_m r) dy_1 \\ & + \frac{i(-1)^{m+1}}{2D_2} \eta_m \int_{-\infty}^{+\infty} f_1(y_1) \frac{\partial^2}{\partial x_1 \partial x_2} H_0^{(1)}(i\lambda_m r) dy_1, \end{aligned}$$

$$u_1^{(4)}=-\frac{i}{2\lambda_4^2}\int\limits_{-\infty}^{+\infty}\left[2\frac{\partial^3}{\partial x_1^2\partial x_2}H_0^{(1)}(\lambda_4 r)+\lambda_4^2\frac{\partial}{\partial x_2}H_0^{(1)}(i\lambda_4 r)\right]f_1(y_1)dy_1$$
$$-\frac{i}{2\lambda_4^2\mu}\int\limits_{-\infty}^{+\infty}f_2(y_1)\frac{\partial^2}{\partial x_1\partial x_2}H_0^{(1)}(i\lambda_4 r)dy_1,$$

$$u_2^{(4)}=\frac{i}{2\lambda_4^2}\int\limits_{-\infty}^{+\infty}\left[-\frac{\partial^3}{\partial x_1^3}H_0^{(1)}(i\lambda_4 r)+\lambda_4^2\frac{\partial^3}{\partial x_1\partial x_2^2}H_0^{(1)}(i\lambda_4 r)\right]f_1(y_1)dy_1$$
$$-\frac{i}{2\lambda_4^2\mu}\int\limits_{-\infty}^{+\infty}f_2(y_1)\frac{\partial^2}{\partial x_1^2}H_0^{(1)}(i\lambda_4 r)dy_1.$$

Solution of Problem 3 for a half-plane

A solution is sought in the form (6),(8). Keeping in mind BCs, after passing to the limit, as $x_2\to 0$, we get the following system of algebraic equations

$$-2\xi^2\sum_{m=1}^{3}\frac{r_m\alpha_m}{\lambda_m^2}-(r_4^2+\xi^2)\alpha_2^{(4)}=\frac{i\xi\widehat{f}_1}{\mu},\qquad \sum_{m=1}^{3}\frac{r_m\alpha_m}{\lambda_m^2}+\alpha_2^{(4)}=\widehat{f}_2,$$
$$\sum_{m=1}^{3}B_m r_m\alpha_m=-\widehat{f}_3,\qquad \sum_{m=1}^{3}C_m r_m\alpha_m=-\widehat{f}_4,\qquad i\xi\alpha_1^{(4)}-r_4\alpha_2^{(4)}=0.$$

From here we get

$$\begin{gathered}\sum_{m=1}^{3}\frac{r_m\alpha_m}{\lambda_m^2}=-\frac{1}{\lambda_4^2}\left[\frac{i\xi\widehat{f}_1}{\mu}+(\xi^2+r_4^2)\widehat{f}_2\right],\\ \sum_{m=1}^{3}B_m r_m\alpha_m=-\widehat{f}_3,\quad \sum_{m=1}^{3}C_m r_m\alpha_m=-\widehat{f}_4,\end{gathered}\tag{11}$$

It is easily seen that the determinant of system (11) has the form

$$D_3=-\frac{\omega^2 d r_1 r_2 r_3}{\lambda_1^2\lambda_2^2\lambda_3^2\delta_1\delta_2\delta_3}[a_1a_2-a_{12}a_{21}](\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_3^2)(\lambda_2^2-\lambda_3^2)=r_1r_2r_3D_2\neq 0.$$

By elementary calculation, from (11) we obtain

$$\alpha_m=\frac{(-1)^m}{r_m D_2}\left\{\frac{\eta_m}{\lambda_4^2}\left[\frac{i\xi\widehat{f}_1}{\mu}+(\xi^2+r_4^2)\widehat{f}_2\right]+c_m\widehat{f}_3-b_m\widehat{f}_4\right\},$$
$$\alpha_1^{(4)}=\frac{r_4}{\lambda_4^2}\left[\frac{\widehat{f}_1}{\mu}-2i\xi\widehat{f}_2\right],\quad \alpha_2^{(4)}=\frac{1}{\lambda_4^2}\left[\frac{i\xi\widehat{f}_1}{\mu}+2\xi^2\widehat{f}_2\right].$$

Finally we have

$$\varphi_m = \frac{i}{2D_2\lambda_4^2}\int\limits_{-\infty}^{+\infty}\eta_m\frac{\partial}{\partial x_1}H_0^{(1)}(i\lambda_m r)\frac{f_1(y_1)}{\mu}dy_1$$
$$-\frac{i}{2D_2\lambda_4^2}\int\limits_{-\infty}^{+\infty}\eta_m\left(2\frac{\partial^2}{\partial x_1^2}H_0^{(1)}(i\lambda_m r)+\lambda_4^2H_0^{(1)}(i\lambda_m r)\right)f_2(y_1)dy_1$$
$$+\frac{i}{2D_2}\int\limits_{-\infty}^{+\infty}[C_mf_3(y_1)-b_mf_4(y_1)]H_0^{(1)}(i\lambda_m r)dy_1,$$

$$u_1^{(4)} = \frac{i}{2\lambda_4^2}\int\limits_{-\infty}^{+\infty}\left[\frac{\partial^2}{\partial x_2^2}H_0^{(1)}(i\lambda_4 r)\frac{f_1(y_1)}{\mu}-2\frac{\partial^3}{\partial x_1\partial x_2^2}H_0^{(1)}(i\lambda_4 r)f_2(y_1)\right]dy_1,$$
$$u_2^{(4)} = \frac{i}{2\lambda_4^2}\int\limits_{-\infty}^{+\infty}\left[-\frac{\partial^2}{\partial x_1\partial x_2}H_0^{(1)}(i\lambda_4 r)\frac{f_1(y_1)}{\mu}+2\frac{\partial^3}{\partial x_2\partial x_1^2}H_0^{(1)}(i\lambda_4 r)f_2(y_1)\right]dy_1,$$

Appendix 1. A Representation of Regular Solutions

Theorem 2. *If $\boldsymbol{U} := (\boldsymbol{u}, p_1, p_2)$ is a regular solution of the homogeneous system (1), then $\boldsymbol{u}$, $div\boldsymbol{u}$, p_1 and p_2 satisfy the equations*

$$\begin{aligned}&(\Delta+\lambda_1^2)(\Delta+\lambda_2^2)(\Delta+\lambda_3^2)(\Delta+\lambda_4^2)\mathbf{u}=0,\\&(\Delta+\lambda_1^2)(\Delta+\lambda_2^2)(\Delta+\lambda_3^2)div\mathbf{u}=0,\\&(\Delta+\lambda_1^2)(\Delta+\lambda_2^2)(\Delta+\lambda_3^2)p_j=0,\quad j=1,2.\end{aligned}\tag{12}$$

where λ_j^2, $j = 1, 2, 3$, are roots of equation (7).

Proof. Let $\mathbf{U} = (\mathbf{u}, p_1, p_2)$ be a regular solution of the equations (1). Upon taking the divergence operation, from (1) we get

$$\begin{aligned}&(\mu_0\Delta+\rho\omega^2)div\mathbf{u}-\beta_1\Delta p_1-\beta_2\Delta p_2=0,\quad \mu_0=\lambda+2\mu,\\&i\omega\beta_1 div\mathbf{u}+(k_1\Delta+a_1)p_1+(k_{12}\Delta+a_{12})p_2=0,\\&i\omega\beta_2 div\mathbf{u}+(k_{21}\Delta+a_{21})p_1+(k_2\Delta+a_2)p_2=0,\end{aligned}$$

Rewrite the latter system as follows

$$D(\Delta)\Psi := \begin{pmatrix}\mu_0\Delta+\rho\omega^2 & -\beta_1\Delta & -\beta_2\Delta\\ i\omega\beta_1 & k_1\Delta+a_1 & k_{12}\Delta+a_{12}\\ i\omega\beta_2 & k_{21}\Delta+a_{21} & k_2\Delta+a_2\end{pmatrix}\Psi=0,\tag{13}$$

where $\Psi = (div\mathbf{u}, p_1, p_2)$.

By the direct calculation, we get

$$det\mathbf{D} = \mu_0\alpha_0(\Delta + \lambda_1^2)(\Delta + \lambda_2^2)(\Delta + \lambda_3^2),$$

Clearly, from system (13) it follows that

$$\begin{aligned}&(\Delta + \lambda_1^2)(\Delta + \lambda_2^2)(\Delta + \lambda_3^2)div\mathbf{u} = 0,\\ &(\Delta + \lambda_1^2)(\Delta + \lambda_2^2)(\Delta + \lambda_3^2)p_j = 0, \quad j = 1, 2.\end{aligned} \tag{14}$$

Further, applying the operator $(\Delta + \lambda_1^2)(\Delta + \lambda_2^2)(\Delta + \lambda_3^2)$ to equation (1), and using the last relations we obtain

$$(\Delta + \lambda_1^2)(\Delta + \lambda_2^2)(\Delta + \lambda_3^2)(\Delta + \lambda_4^2)\mathbf{u} = 0. \tag{15}$$

The last formulas (14),(15) prove the theorem.

Theorem 3. T*he regular solution* $\boldsymbol{U} = (\boldsymbol{u}, p_1, p_2)$ *of system (1) admits in the domain of regularity a representation*

$$\mathbf{U} = (\overset{\mathbf{1}}{\mathbf{u}} + \overset{\mathbf{2}}{\mathbf{u}}, p_1, p_2), \tag{16}$$

where $\overset{\mathbf{1}}{\mathbf{u}}$, *and* $\overset{\mathbf{2}}{\mathbf{u}}$ *are the regular vectors, satisfying the conditions*

$$\begin{aligned}&(\Delta + \lambda_1^2)(\Delta + \lambda_2^2)(\Delta + \lambda_3^2)\overset{\mathbf{1}}{\mathbf{u}} = 0, \quad rot\overset{\mathbf{1}}{\mathbf{u}} = 0,\\ &(\Delta + \lambda_4^2)\overset{\mathbf{2}}{\mathbf{u}} = 0, \quad div\overset{\mathbf{2}}{\mathbf{u}} = 0.\end{aligned}$$

Proof. Let $\mathbf{U} = (\mathbf{u}, p_1, p_2)$ be a regular solution of system (1). Using the identity

$$\Delta\mathbf{w} = graddiv\mathbf{w} - rotrot\mathbf{w}, \tag{17}$$

from Eq.(1) we obtain

$$\mathbf{u} = -\frac{\mu_0}{\rho\omega^2}graddiv\mathbf{u} + \frac{\mu}{\rho\omega^2}rotrot\mathbf{u} + \frac{1}{\rho\omega^2}grad(\beta_1 p_1 + \beta_2 p_2),$$

Let

$$\overset{\mathbf{1}}{\mathbf{u}} := -\frac{\mu_0}{\rho\omega^2}graddiv\mathbf{u} + \frac{1}{\rho\omega^2}grad(\beta_1 p_1 + \beta_2 p_2), \tag{18}$$

$$\overset{\mathbf{2}}{\mathbf{u}} := \frac{\mu}{\rho\omega^2}rotrot\mathbf{u}. \tag{19}$$

Clearly

$$\mathbf{u} = \overset{\mathbf{1}}{\mathbf{u}} + \overset{\mathbf{2}}{\mathbf{u}}, \quad rot\overset{\mathbf{1}}{\mathbf{u}} = 0, \quad div\overset{\mathbf{2}}{\mathbf{u}} = 0. \tag{20}$$

Using the identity $\Delta\overset{\mathbf{2}}{\mathbf{u}} = -rotrot\overset{\mathbf{2}}{\mathbf{u}}$, from (19) we obtain

$$(\Delta + \lambda_4^2)\overset{\mathbf{2}}{\mathbf{u}} = 0. \tag{21}$$

Taking into account the relations (14),(15),(18) and (19) we can easily prove the following

Theorem 4. *In the domain of regularity the regular solution* $\boldsymbol{U} = (\boldsymbol{u}, p_1, p_2) \in C^2(D)$ *of system (1) is represented as the sum*

$$\begin{aligned} &\boldsymbol{u}(\boldsymbol{x}) = -grad \sum_{m=1}^{3} \frac{\varphi_m(\boldsymbol{x})}{\lambda_m^2} + \boldsymbol{u}^{(2)}(\boldsymbol{x}), \\ &p_1(\boldsymbol{x}) = \sum_{m=1}^{3} B_m \varphi_m(\boldsymbol{x}), \quad p_2(\boldsymbol{x}) = \sum_{m=1}^{3} C_m \varphi_m(\boldsymbol{x}), \end{aligned} \qquad (22)$$

where

$$\begin{aligned} &(\Delta + \lambda_m^2)\varphi_m(\boldsymbol{x}) = 0, \quad (\Delta + \lambda_4^2)\boldsymbol{u}^{(2)}(\boldsymbol{x}) = 0, \quad div\boldsymbol{u}^{(2)}(\boldsymbol{x}) = 0, \\ &B_m = -\frac{i\omega}{\delta_m}\left[\beta_1(a_2 - k_2\lambda_m^2) - \beta_2(a_{12} - k_{12}\lambda_m^2)\right], \\ &C_m = -\frac{i\omega}{\delta_m}\left[\beta_2(a_1 - k_1\lambda_m^2) - \beta_1(a_{21} - k_{21}\lambda_m^2)\right], \\ &\delta_m = (k_1k_2 - k_{12}k_{21})\lambda_m^4 - k_0\lambda_m^2 + a_1a_2 - a_{12}a_{21}. \end{aligned}$$

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