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CONVERGENCE IN MEASURE OF LOGARITHMIC MEANS OF DOUBLE FOURIER SERIES

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Abstract. We establish condition which guarantees convergence in measure of logarithmic means of the two-dimensional Fourier series.

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Let $\mathbb{T}^2 := [-\pi, \pi)^2$ denote a cube in the 2-dimensional Euclidean space $\mathbb{R}^2$. The elements of $\mathbb{R}^2$ are denoted by (x, y).

The notation $a \lesssim b$ in the paper stands for $a \leq cb$, where c is an absolute constant.

We denote by $L_0(\mathbb{T}^2)$ the Lebesgue space of functions that are measurable and finite almost everywhere on $\mathbb{T}^2$. $\operatorname{mes}(A)$ is the Lebesgue measure of the set $A \subset \mathbb{T}^2$.

We denote by $L_p(\mathbb{T}^2)$ the class of all measurable functions f that are 2π-periodic with respect to all variables and satisfy

$$\|f\|_p := \left(\int\limits_{\mathbb{T}^2} |f|^p \right)^{1/p} < \infty.$$

The $weak - L_1(\mathbb{T}^2)$ space consists of all measurable, 2π-periodic relative to each variable functions f for which

$$\|f\|_{weak-L_1(\mathbb{T}^2)} := \sup_{\lambda} \lambda \operatorname{mes} \left\{ (x, y) \in \mathbb{T}^2 : |f(x, y)| > \lambda \right\} < \infty.$$

Let $f \in L_1(\mathbb{T}^2)$. The Fourier series of f with respect to the trigonometric system is the series

$$S[f] := \sum_{n,m=-\infty}^{+\infty} \widehat{f}(n, m) e^{i(nx+my)},$$

where

$$\widehat{f}(n, m) := \frac{1}{(2\pi)^2} \int\limits_{\mathbb{T}^2} f(x, y) e^{-i(nx+my)} dxdy$$

are the Fourier coefficients of the function f. The rectangular partial sums are defined as follows:

$$S_{NM}(f; x, y) := \sum_{n=-N}^{N} \sum_{m=-M}^{M} \widehat{f}(n, m) e^{i(nx+my)}.$$

In the literature the notion of the Riesz's logarithmic means of a Fourier series is known. The n-th Riesz logarithmic mean of the Fourier series of the integrable function

f is defined by

$$\frac{1}{l_n}\sum_{k=0}^{n}\frac{S_k(f)}{k+1},\quad l_n:=\sum_{k=0}^{n}\frac{1}{k+1},$$

where $S_k(f)$ is the partial sum of its Fourier series. This Riesz's logarithmic means with respect to the trigonometric system has been studied by a lot of authors. We mention for instance the papers of Szász, and Yabuta [13, 15]. This mean with respect to the Walsh, Vilenkin system is discussed by Simon, and Gát [12, 2].

Let $\{q_k : k \geq 0\}$ be a sequence of nonnegative numbers. The Nörlund means for the Fourier series of f are defined by

$$\frac{1}{\sum_{k=0}^{n} q_k}\sum_{k=0}^{n} q_k S_{n-k}(f).$$

If $q_k = \frac{1}{k+1}$, then we get the (Nörlund) logarithmic means:

$$L_n(f;x) := \frac{1}{l_n}\sum_{k=0}^{n}\frac{S_{n-k}(f)}{k+1}. \tag{1}$$

Although, it is a kind of "reverse" Riesz's logarithmic means. In [5] some convergence and divergence properties of the logarithmic means of Walsh-Fourier series of functions in the class of continuous functions, and in the Lebesgue space L are proved.

In one of his last papers [14] Tkebuchava constructed a set of logarithmic summation methods which contains both of the above mentioned logarithmic summation methods as limit cases. Namely, for any integers n, n_0 such that $0 \leq n_0 \leq n$ let Tkebuchava's means T_{n,n_0} be defined by

$$T_{n,n_0}(f;x)$$
$$:= \frac{1}{l(n,n_0)}\left(\sum_{k=0}^{n_0-1}\frac{S_k(f;x)}{n_0-k+1} + S_{n_0}(f;x) + \sum_{k=n_0+1}^{n}\frac{S_k(f;x)}{k-n_0+1}\right),$$

where

$$l(n,n_0) := \sum_{k=0}^{n_0-1}\frac{1}{n_0-k+1} + 1 + \sum_{k=n_0+1}^{n}\frac{1}{k-n_0+1}.$$

It is clear that $l(n,n_0) \asymp \log n$. This summation method includes the Riesz (for $n_0 = 0$) and Nörlund (for $n_0 = n$) logarithmic methods, too.

Define the kernels F_{n,n_0} of Tkebuchava's means by

$$F_{n,n_0} := \frac{1}{l(n,n_0)}\left(\sum_{k=0}^{n_0-1}\frac{D_k}{n_0-k+1} + D_{n_0} + \sum_{k=n_0+1}^{n}\frac{D_k}{k-n_0+1}\right).$$

Tkebuchava [14] gave estimates of kernels. Namely, the following theorem holds.

Theorem T (Tkebuchava). *Let* $0 \leq n_0 \leq n$. *Then*

$$1 + \frac{\log^2(n_0+2)}{\log(n+2)} \lesssim \|F_{n,n_0}\|_{L_1(\mathbb{T})} \lesssim 1 + \frac{\log^2(n_0+2)}{\log(n+2)}.$$

The mixed logarithmic means of double Fourier series are defined by

$$(L_n \circ R_m)(f;x,y) := \frac{1}{l_n l_m} \sum_{i=0}^{n} \sum_{j=0}^{m} \frac{S_{n-i,j}(f;x,y)}{(i+1)(j+1)}.$$

The Nörlund logarithmic means and Riesz logarithmic means of double Fourier series are defined by

$$(L_n \circ L_m)(f;x,y) := \frac{1}{l_n l_m} \sum_{i=0}^{n} \sum_{j=0}^{m} \frac{S_{n-i,m-j}(f;x,y)}{(i+1)(j+1)},$$

$$(R_n \circ R_m)(f;x,y) := \frac{1}{l_n l_m} \sum_{i=0}^{n} \sum_{j=0}^{m} \frac{S_{i,j}(f;x,y)}{(i+1)(j+1)},$$

respectively.

It is evident that

$$(L_n \circ L_m)(f;x,y) = \frac{1}{\pi^2} \int_{\mathbb{T}^2} f(s,t) F_n(x-s) F_m(y-t)\, dsdt,$$

$$(R_n \circ R_m)(f;x,y) = \frac{1}{\pi^2} \int_{\mathbb{T}^2} f(s,t) G_n(x-s) G_m(y-t)\, dsdt$$

and

$$(L_n \circ R_m)(f;x,y) = \frac{1}{\pi^2} \int_{\mathbb{T}^2} f(s,t) F_n(x-s) G_m(y-t)\, dsdt,$$

where

$$F_n(u) := \frac{1}{l_n} \sum_{i=0}^{n} \frac{D_{n-i}(u)}{i+1}, G_n(u) := \frac{1}{l_n} \sum_{i=0}^{n} \frac{D_i(u)}{i+1}.$$

Let $L_Q = L_Q(\mathbb{T}^2)$ be the Orlicz space ([10], Ch 2) generated by Young function Q, i.e. Q is a convex continuous even function such that $Q(0) = 0$ and

$$\lim_{u\to+\infty} \frac{Q(u)}{u} = +\infty, \quad \lim_{u\to 0} \frac{Q(u)}{u} = 0.$$

This space is endowed with the norm

$$\|f\|_{L_Q(\mathbb{T}^2)} = \inf\{k > 0 : \int_{\mathbb{T}^2} Q(|f|/k) \le 1\}.$$

In particular, if $Q(u) = u \log^{\beta}(1+u)$ $(u, \beta > 0)$, then the corresponding space will be denoted by $L \log^{\beta} L(\mathbb{T}^2)$.

The rectangular partial sums of double Fourier series $S_{n,m}(f;x,y)$ of the function $f \in L_p(\mathbb{T}^2), 1 < p < \infty$ converge in L_p norm to the function f, as $n \to \infty$ [16]. In the case $L_1(\mathbb{T}^2)$ this result does not hold . But for one dimensional case and for $f \in L_1(\mathbb{T})$, the operator $S_n(f)$ is of weak type (1,1) [17]. This estimate implies convergence of $S_n(f;x)$ in measure on $\mathbb{T}$ to the function $f \in L_1(\mathbb{T})$. However, for double Fourier series this result does not hold [9, 11]. Moreover, it is proved that quadratical partial sums $S_{n,n}(f;x,y)$ of double Fourier series do not converge in two-dimensional measure on $\mathbb{T}^2$ even for functions from Orlicz spaces wider than the Orlicz space $L\log L(\mathbb{T}^2)$. On the other hand, it is well-known that if the function $f \in L\log L(\mathbb{T}^2)$, then rectangular partial sums $S_{n,m}(f;x,y)$ converge in measure on $\mathbb{T}^2$.

Classical regular summation methods often improve the convergence of Fourier seeries. For instance, the Fejér means of the double Fourier series of the function $f \in L_1(\mathbb{T}^2)$ converge in $L_1(\mathbb{T}^2)$ norm to the function f [16]. These means present the particular case of the Nörlund means.

It is well known that the method of Nörlund logarithmic means of double Fourier series is weaker than the Cesáro method of any positive order. In [7] it is proved, that these means of double Fourier series in general do not converge in two-dimensional measure on $\mathbb{T}^2$ even for functions from Orlicz spaces wider than Orlicz space $L\log L(\mathbb{T}^2)$. Thus, not all classic regular summation methods can improve the convergence in measure of double Fourier series.

The results for summability of logarithmic means of Walsh-Fourier series can be found in [3, 4, 6, 5, 13, 15].

In [7] the mixed logarithmic means $(L_n \circ R_m)$ of rectangular partial sums multiple Fourier series are considered and it is proved that these means are acting from space $L(\mathbb{T}^2)$ into space $weak - L_1(\mathbb{T}^2)$. This fact implies that mixed logarithmic means of rectangular partial sums of double Fourier series converge in measure. In particular, the following is true.

Theorem GG1(Goginava, Gogoladze). *Let $f \in L_1(\mathbb{T}^2)$. Then*

$$(R_n \circ L_m)(f;x,y) \to f \text{ in measure on } \mathbb{T}^2, \text{ as } n,m \to \infty.$$

Theorem GG2 (Goginava, Gogoladze). *Let $f \in L\log L(\mathbb{T}^2)$. Then*

$$(L_n \circ L_m)(f;x,y) \to f \text{ in measure on } \mathbb{T}^2, \text{ as } n,m \to \infty.$$

Theorem GG3 (Goginava, Gogoladze). *Let $L_Q(\mathbb{T}^2)$ be an Orlicz space, such that*

$$L_Q\left(\mathbb{T}^d\right) \nsubseteq L\log L\left(\mathbb{T}^2\right).$$

Then the set of the functions from the Orlicz space $L_Q(\mathbb{T}^2)$ with logarithmic means $(L_n \circ L_m)(f)$ of rectangular partial sums of double Fourier series convergent in measure on $\mathbb{T}^2$ is of first Baire category in $L_Q(\mathbb{T}^2)$.

For any integers n, n_0, m such that $0 \leq n_0 \leq n$ we put

$$(T_{n,n_0} \circ L_m)(f;x,y) = f * (F_{n,n_0} \times F_m).$$

It is easy to show that

$$\left(T_{n,n_0} \circ L_m\right)(f;x,y) = \frac{1}{\pi^2} \int\limits_{\mathbb{T}^2} f(s,t)\, F_{n,n_0}(x-s)\, F_m(y-t)\, dsdt.$$

This summation method includes the $(R_n \circ L_m)$ (for $n_0 = 0$) and $(L_n \circ L_m)$ (for $n_0 = n$) methods, too.

On the basis of the above facts we can formulate the following problem:

Let $f \in L_1(\mathbb{T}^2)$. What condition on the $n_0 = n_0(n)$ ensure the convergence in measure on $\mathbb{T}^2$ of the $(T_{n,n_0} \circ L_m)$ means of the two-dimensional trigonometric Fourier series?

A solution of this problem is given in

Theorem 1. *a)Let $f \in L_1(\mathbb{T}^2)$ and*

$$\log n_0(n) = O\left(\sqrt{\log n}\right).$$

Then

$$\left(T_{n,n_0} \circ L_m\right)(f;x,y) \to f \text{ in measure on } \mathbb{T}^2, \quad \text{as} \quad n,m \to \infty.$$

b) Let

$$\overline{\lim_{n\to\infty}} \frac{\log n_0(n)}{\sqrt{\log n}} = \infty.$$

Then the set of the functions from the space $L_1(\mathbb{T}^2)$ with logarithmic means $(T_{n,n_0} \circ L_m)(f)$ of rectangular partial sums of double Fourier series convergent in measure on $\mathbb{T}^2$ is of first Baire category in $L_1(\mathbb{T}^2)$.

In order to prove Theorem we apply the reasoning of ([1], Ch. 1) formulated as the following proposition in a particular case.

Theorem G. *Let $H : L_1(\mathbb{T}^2) \to L_0(\mathbb{T}^2)$ be a linear continuous operator, which commutes with family of translations $\mathcal{E}$, i. e. $\forall E \in \mathcal{E}$ $\quad \forall f \in L_1(\mathbb{T}^2)$ $\quad HEf = EHf$. Let $\|f\|_{L_1(\mathbb{T}^2)} = 1$ and $\lambda > 1$. Then for any $1 \le r \in \mathbb{N}$ under condition $mes\{(x,y) \in \mathbb{T}^2 : |Hf| > \lambda\} \ge \frac{1}{r}$ there exist $E_1,...,E_r,E'_1,...,E'_r \in \mathcal{E}$ and $\varepsilon_i = \pm 1$, $\quad i = 1,...,r$ such that*

$$\mathrm{mes}\{(\mathrm{x},\mathrm{y}) \in \mathbb{T}^2 : \left|\mathrm{H}\left(\sum_{\mathrm{i}=1}^{\mathrm{r}} \varepsilon_{\mathrm{i}} \mathrm{f}(\mathrm{E_i x}, \mathrm{E'_i y})\right)\right| > \lambda\} \ge \frac{1}{8}.$$

Theorem GGT (Gát, Goginava, Tkebuchava). *Let $\{H_m\}_{m=1}^{\infty}$ be a sequence of linear continuous operators, acting from the space $L_1(\mathbb{T}^2)$ into the space $L_0(\mathbb{T}^2)$. Suppose that there exists the sequence of functions $\{\xi_k\}_{k=1}^{\infty}$ from the unit ball $S(0,1)$ of space $L_1(\mathbb{T}^2)$, sequences of integers $\{m_k\}_{k=1}^{\infty}$ and $\{\nu_k\}_{k=1}^{\infty}$ increasing to infinity such that*

$$\varepsilon_0 = \inf_k \mathrm{mes}\{(x,y) \in \mathbb{T}^2 : |H_{m_k}\xi_k(x,y)| > \nu_k\} > 0.$$

Then K - the set of functions f from the space $L_1(\mathbb{T}^2)$, for which the sequence $\{H_m f\}$ converges in measure to an a. e. finite function is of first Baire category in the space $L_1(\mathbb{T}^2)$.

The proof of Lemma GGT can be found in [3].

Set

$$\alpha_{km} := \frac{\pi(12k+1)}{6(m+1/2)}, \beta_{km} := \frac{\pi(12k+5)}{6(m+1/2)}, \gamma_m := \frac{\pi}{6(m+1/2)},$$

$$J_m := \bigcup_{k=1}^{\left[\frac{\sqrt{m+1}-5}{12}\right]} [\alpha_{km}+\gamma_m, \beta_{km}-\gamma_m].$$

Lemma T (Tkebuchava). *Let* $0 \le z \le \gamma_n$ *and* $x \in J_n$. *Then*

$$F_{n,n_0}(x-z) \gtrsim \frac{\log(n_0+2)}{x\log(n+2)}.$$

The proof of Lemma T can be found in [6].

Proof of Theorem 1. a) In [8] it is proved that the one dimensional operator $L_m(f) := f * F_m$ (see (1)) is of weak type $(1,1)$, i. e. for $f \in L_1(\mathbb{T}^1)$ we have

$$\|L_m(f)\|_{weak-L_1(\mathbb{T}^1)} \lesssim \|f\|_{L_1(\mathbb{T}^1)}. \tag{2}$$

On the other hand, Tkebuchava in [14] proved that

$$\sup_n \|F_{n,n_0}\|_{L_1(\mathbb{T})} < \infty$$

when

$$\log n_0 = O\left(\sqrt{\log n}\right). \tag{3}$$

Set

$$\Omega := \left\{(x,y) \in \mathbb{T}^{\mathbf{2}} : |(\mathbf{T_{n,n_0}} \circ \mathbf{L_m})(\mathbf{f},\mathbf{x},\mathbf{y})| > \lambda\right\}.$$

Then from (2) and (3) we have

$$\begin{aligned} &\lambda \operatorname{mes}(\Omega) \\ = &\ \lambda \int_{\mathbb{T}^2} \mathbb{I}_\Omega(x,y)\,dxdy = \lambda \int_{\mathbb{T}} \left(\int_{\mathbb{T}} \mathbb{I}_\Omega(x,y)\,dy\right) dx \\ \lesssim &\ \|(f * F_{n,n_0})(f)\|_{L_1(\mathbb{T}^2)} \lesssim \|f\|_{L_1(\mathbb{T}^2)}, \end{aligned} \tag{4}$$

where $\mathbb{I}_{\mathbb{E}}$ is a characteristic function of the set E.

By virtue of standart argument (see [17]) we can prove the validity of part a) from the estimation (4).

Now, we prove part b). Let

$$\overline{\lim_{n\to\infty}} \frac{\log n_0(n)}{\sqrt{\log n}} = \lim_{k\to\infty} \frac{\log n_0(n_k)}{\sqrt{\log n_k}} = \infty.$$

By Lemma GGT the proof of Theorem will be complete if we show that there exists for the sequences of integers $\{n_k : k \ge 1\}$ and $\{\nu_k : k \ge 1\}$ increasing to infinity, and a sequence of functions $\{\xi_k : n \geq 1\}$ from the unit bull $S(0,1)$ of space $L_1(\mathbb{T}^2)$, such that for all n

$$\text{mes}\{(x,y) \in \mathbb{T}^2 : \left|\left(T_{n_k,n_0(n_k)} \circ L_{n_k}\right)(\xi_k; x,y)\right| > \nu_k\} \geq \frac{1}{8}. \tag{5}$$

First, we prove that

$$\text{mes}\left\{(x,y) \in \mathbb{T}^2 : \left|\left(T_{n_k,n_0(n_k)} \circ L_{n_k}\right)\left(\frac{\mathbb{I}_{[0,\gamma_{n_k}]^2}}{\gamma_{n_k}^2}; x,y\right)\right| \gtrsim n_k^{3/2}\right\} \tag{6}$$
$$\gtrsim \frac{\log^2 n_0(n_k)}{n_k^{3/2}\log n_k}.$$

From Lemma T we have

$$\begin{aligned}
&\left(T_{n_k,n_0(n_k)} \circ L_{n_k}\right)\left(\frac{\mathbb{I}_{[0,\gamma_{n_k}]^2}}{\gamma_{n_k}^2}; x,y\right) \\
= &\frac{1}{\gamma_{n_k}^2}\frac{1}{\pi^2}\int\limits_{[0,\gamma_{n_k}]^2} F_{n_k,n_0(n_k)}(x-u)\,F_{n_k}(y-v)\,dudv \\
\gtrsim &\frac{\log n_0(n_k)}{\log n_k}\frac{1}{xy}, (x,y) \in J_{n_k} \times J_{n_k}.
\end{aligned}$$

Set

$$s_{i,n_k} := \frac{\sqrt{n_k}\log n_0(n_k)}{i\log n_k}.$$

Then we can write

$$\begin{aligned}
&\text{mes}\left\{(x,y) \in \mathbb{T}^2 : \left|\left(T_{n_k,n_0(n_k)} \circ L_{n_k}\right)\left(\frac{\mathbb{I}_{[0,\gamma_{n_k}]^2}}{\gamma_{n_k}^2}; x,y\right)\right| \gtrsim n_k^{3/2}\right\} \\
\geq\ &\text{mes}\left\{(x,y) \in J_{n_k} \times J_{n_k} : \left|\left(T_{n_k,n_0(n_k)} \circ L_{n_k}\right)\left(\frac{\mathbb{I}_{[0,\gamma_{n_k}]^2}}{\gamma_{n_k}^2}; x,y\right)\right| \gtrsim n_k^{3/2}\right\} \\
\geq\ &\text{mes}\left\{(x,y) \in J_{n_k} \times J_{n_k} : \frac{\log n_0(n_k)}{\log n_k}\frac{1}{xy} \gtrsim n_k^{3/2}\right\} \\
=\ &\text{mes}\left\{(x,y) \in J_{n_k} \times J_{n_k} : y \lesssim \frac{\log n_0(n_k)}{x n_k^{3/2}\log n_k}\right\} \\
\gtrsim\ &\frac{1}{n_k^2}\sum_{i=1}^{\left[\frac{\sqrt{n_0(n_k)+1}-5}{12}\right]}\sum_{l=1}^{s_{i,n_k}} \\
=\ &c\sum_{i=1}^{\left[\frac{\sqrt{n_0(n_k)+1}-5}{12}\right]}\frac{\sqrt{n_k}\log n_0(n_k)}{i n_k^2\log n_k} \\
\gtrsim\ &\frac{\log^2 n_0(n_k)}{n_k^{3/2}\log n_k},
\end{aligned}$$

Hence (6) is proved.

Then by the virtue of Theorem G there exists $E_1, ..., E_{r_k}, E'_1, ..., E'_{r_k} \in \mathcal{E}$ and $\varepsilon_1, ..., \varepsilon_{r_k} = \pm 1$ such that

$$\begin{aligned} \operatorname{mes}\{(x,y) \quad &\in \quad \mathbb{T}^2 : \left| \sum_{i=1}^{r_k} \varepsilon_i \left(T_{n_k, n_0(n_k)} \circ L_{n_k} \right) \left(\frac{\mathbb{I}_{[0,\gamma_{n_k}]^2}}{\gamma_{n_k}^2}; E_i x, E'_i y \right) \right| \\ &\gtrsim \quad n_k^{3/2} \} > \frac{1}{8}, \end{aligned} \tag{7}$$

where

$$r_k \sim \frac{n_k^{3/2} \log n_k}{\log^2 n_0 (n_k)}.$$

Denote

$$\nu_k = \frac{\log^2 n_0 (n_k)}{\log n_k}$$

and

$$\xi_k (x,y) = \frac{1}{r_k} \sum_{i=1}^{r_k} \varepsilon_i \frac{\mathbb{I}_{[0,\gamma_{n_k}]^2} (E_i x, E'_i y)}{\gamma_{n_k}^2}.$$

Thus, from (7) we obtain (5).

Finally, we prove that $\xi_k \in S(0,1)$. Indeed,

$$\|\xi_k\|_{L_1(\mathbb{T}^2)} \leq \frac{1}{r_k} \sum_{i=1}^{r_k} \frac{\left\| \mathbb{I}_{[0,\gamma_{n_k}]^2} \right\|_{L_1(\mathbb{T}^2)}}{\gamma_{n_k}^2} \leq 1.$$

Hence, $\xi_k \in S(0,1)$, and Theorem is proved.

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