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THE BOUNDARY VALUE PROBLEMS IN THE FULL COUPLED THEORY OF ELASTICITY FOR PLANE WITH DOUBLE POROSITY WITH A CIRCULAR HOLE[1]

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Abstract. The purpose of this paper is to consider two-dimensional version of the full coupled theory of elasticity for solids with double porosity and to solve explicitly the Dirichlet and Neumann BVPs of statics in the full coupled theory for an elastic plane with a circular hole. The explicit solutions of these BVPs are represented by means of absolutely and uniformly convergent series. The questions on the uniqueness of a solutions of the problems are established.

Keywords and phrases: Double porosity, explicit solution,elastic plane with circular hole, absolutely and uniformly convergent series.

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Introduction

Many geothermal fields are naturally fractured systems. Classic double porosity models the flow between matrix and fractures, under the hypothesis that petrophysical properties are uniform in each medium. Fractures have the largest permeability and drive the fluid toward the wells. The matrix, with smaller permeability, only acts as a source of fluid for the fractures. Double porosity models can be classified as special cases of this general theoretical concept, applicable to all class reservoirs. The matrix blocks surrounded by fractures can have several geometries and any size. Fractures have very little storage, but provide the high permeability conduits to drive the fluid toward the wells. Matrix blocks have higher porosity and constitute the largest storage, but have smaller permeability, acting only as a source of stationary fluid for the fractures.

A theory of consolidation with double porosity has been proposed by Aifantis. This theory unifies a model proposed by Biot for the consolidation of deformable single porosity media with a model proposed by Barenblatt for seepage in undeformable media with two degrees of porosity. In a material with two degrees of porosity, there are two pore systems, the primary and the secondary. For example in a fissured rock (i.e.a mass of porous blocks separated from each other by an interconnected and continuously distributed system of fissures) most of the porosity is provided by the pores of the blocks or primary porosity, while most of permeability is provided by the fissures or the secondary porosity.

The physical and mathematical foundations of the theory of double porosity were considered in the papers [1]-[3]. In part I of a series of paper on the subject, R. K. Wilson and E. C. Aifantis [2] gave detailed physical interpretations of the phenomenological

[1]This paper dedicated to our teacher to the 85^{th} birth anniversary of professor Mikheil Basheleishvili

coefficients appearing in the double porosity theory. They also solved several representative boundary value problems. In part II of this series, uniqueness and variational principles were established by D. E. Beskos and E. C. Aifantis [3] for the equations of double porosity, while in part III Khaled, Beskos and Aifantis [4] provided a related finite element to consider the numerical solution of Aifantis' equations of double porosity (see [2],[3],[4] and the references cited therein). The basic results and the historical information on the theory of porous media were summarized by Boer [5].

However, Aifantis' quasi-static theory ignored the cross-coupling effect between the volume change of the pores and fissures in the system. The cross-coupled terms were included in the equations of conservation of mass for the pore and fissure fluid and in Darcy's law for solids with double porosity by several authors [5,9]. In [10] the full coupled linear theory of elasticity for solids with double porosity is considered. Four spatial cases of the dynamical equations are considered. The fundamental solutions are constructed by means of elementary functions and the basic properties of the fundamental solutions are established. The fundamental solution of quasi-static equations of the linear theory elasticity for double porosity solids is constructed and basic properties are established in [11]. In [12-15] the explicit solutions of the problems of porous elastostatics for an elastic circle and for the plane with a circular hole are constructed, the uniqueness theorems for regular solutions are proved and the numerical results are given for boundary value problems. Explicit solutions of the BVPs of the theory of consolidation with double porosity for the half-plane and half-space are considered in [16,17].

The purpose of this paper is to consider two-dimensional version of the full coupled theory of elasticity for solids with double porosity and to solve explicitly the Dirichlet and Neumann BVPs of statics in the full coupled theory for an elastic plane with a circular hole. The explicit solutions of these BVPs are represented by means of absolutely and uniformly convergent series. The questions on the uniqueness of a solutions of the problems are established.

Basic equations and boundary value problems

Let D be a plane with a circular hole. Let R be the radius of a circle with the boundary S centered at point $O(0,0)$. Let us assume that the domain D is filled with an isotropic material with double porosity.

The system of homogeneous equations in the full coupled linear equilibrium theory of elasticity for materials with double porosity can be written as follows [6,10]

$$\mu\Delta\mathbf{u} + (\lambda+\mu) graddiv\mathbf{u} - grad(\beta_1 p_1 + \beta_2 p_2) = 0, \tag{1}$$

$$\begin{aligned} &(k_1\Delta - \gamma)p_1 + (k_{12}\Delta + \gamma)p_2 = 0, \\ &(k_{21}\Delta + \gamma)p_1 + (k_2\Delta - \gamma)p_2 = 0, \end{aligned} \tag{2}$$

where $\mathbf{u} = \mathbf{u}(u_1, u_2)^T$ is the displacement vector in a solid, p_1 and p_2 are the pore and fissure fluid pressures respectively. β_1 and β_2 are the effective stress parameters, $\gamma > 0$ is the internal transport coefficient and corresponds to fluid transfer rate with respect to the intensity of flow between the pore and fissures, λ, μ, are constitutive coefficients, $k_j = \dfrac{\kappa_j}{\mu'}$, $k_{12} = \dfrac{\kappa_{12}}{\mu'}$, $k_{21} = \dfrac{\kappa_{21}}{\mu'}$. μ' is the fluid viscosity, κ_1 and κ_2 are the macroscopic intrinsic permeabilities associated with matrix and fissure porosity, respectively, κ_{12} and κ_{21} are the cross-coupling permeabilities for fluid flow at the interface between the matrix and fissure phases, Δ is the 2D Laplace operator. Throughout this article it is assumed that $\beta_1^2 + \beta_2^2 > 0$, and the superscript "T" denotes transposition.

Introduce the definition of a regular vector-function.

Definition. A vector-function $\mathbf{U(x)} = (u_1, u_2, p_1, p_2)$ defined in the domain D is called regular if it has integrable continuous second derivatives in D, and $\mathbf{U(x)}$ itself and its first order derivatives are continuously extendable at every point of the boundary of D, i.e., $\mathbf{U(x)} \in C^2(D) \bigcap C^1(\overline{D})$; $\mathbf{x} \in D$, $\mathbf{x} = (x_1, x_2)$. Note that in the domain D the vector $\mathbf{U(x)}$ additionally has to satisfy certain conditions at infinity.

Note that system (2) would be considered separately. Further we assume that p_j is known, when $\mathbf{x} \in D$.

Supposing

$$\begin{pmatrix} p_1(\mathbf{x}) \\ p_2(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} k_2\Delta - \gamma & -(k_{12}\Delta + \gamma) \\ -(k_{21}\Delta + \gamma) & k_1\Delta - \gamma \end{pmatrix} \boldsymbol{\psi}(\mathbf{x}),$$

where $\boldsymbol{\psi} = (\psi_1, \psi_2)$ is a four times differentiable vector function, we can write the system (2) as

$$(\Delta + \lambda_1^2)\Delta\psi_j(\mathbf{x}) = 0. \tag{3}$$

With the help of (3) we find the solution of system (2) in the form

$$p_1(\mathbf{x}) = \varphi(\mathbf{x}) + A_1\varphi_1(\mathbf{x}), \quad p_2(\mathbf{x}) = \varphi(\mathbf{x}) + \varphi_1(\mathbf{x}), \tag{4}$$

where

$$\Delta\varphi = 0, \quad (\Delta + \lambda_1^2)\varphi_1 = 0, \quad A_1 = \frac{\gamma - k_{12}\lambda_1^2}{\gamma + k_1\lambda_1^2} = -\frac{k_2 + k_{12}}{k_1 + k_{21}},$$

$$\lambda_1 = i\sqrt{\frac{\gamma k_0}{k_1k_2 - k_{12}k_{21}}} = i\lambda_0,\ i = \sqrt{-1},\ k_0 = k_1 + k_2 + k_{12} + k_{21};$$

$$k_1 > 0, \quad k_2 > 0, \quad \gamma > 0, \quad k_1k_2 - k_{12}k_{21} > 0, \quad k_0 > 0.$$

Let us substitute the expression $\beta_1 p_1 + \beta_2 p_2$ into (1) and let us search the particular solution of the following nonhomogeneous equation

$$\mu\Delta\mathbf{u} + (\lambda + \mu) graddiv\mathbf{u} = grad[(\beta_1 + \beta_2)\varphi + (A_1\beta_1 + \beta_2)\varphi_1].$$

It is well-known that a general solution of the last equation is presented in the form

$$\mathbf{u}(\mathbf{x}) = \mathbf{v}(\mathbf{x}) + \mathbf{v}_0(\mathbf{x}), \tag{5}$$

where $\mathbf{v}(\mathbf{x})$ is a general solution of the equation

$$\mu\Delta\mathbf{v} + (\lambda + \mu) graddiv\mathbf{v} = 0, \tag{6}$$

and $\mathbf{v}_0(\mathbf{x})$ is a particular solution of the nonhomogeneous equation

$$\mathbf{v}_0(\mathbf{x}) = \frac{1}{\lambda + 2\mu} grad \left[(\beta_1 + \beta_2)\varphi_0 - \frac{\beta_1 A_1 + \beta_2}{\lambda_1^2}\varphi_1 \right], \tag{7}$$

where φ_0 is a biharmonic function $\Delta\Delta\varphi_0 = 0$ and $\Delta\varphi_0 = \varphi$, $\Delta\varphi = 0$.

So it remains to study the problem of finding the functions $p_j(\mathbf{x})$, $j = 1, 2$.

We consider only the exterior boundary value problems. The interior one can be treated quite similarly.

The basic BVPs in the full coupled linear equilibrium theory of elasticity for materials with double porosity are formulated as follows.

The Dirichlet BVP problem. Find a regular solution $\mathbf{U}(\mathbf{u}, p_1, p_2)$ to systems (1) and (2) for $\mathbf{x} \in D$ satisfying the following boundary conditions:

$$\mathbf{u} = \mathbf{f}(\mathbf{z}), \quad p_1(\mathbf{z}) = f_3(\mathbf{z}), \quad p_2(\mathbf{z}) = f_4(\mathbf{z}), \quad \mathbf{z} \in S; \tag{8}$$

Note that for the domain D the vector $\mathbf{U(x)}$ additionally has to satisfy the following decay conditions at infinity

$$\mathbf{U}(\mathbf{x}) = o(1), \quad \frac{\partial \mathbf{U}(\mathbf{x})}{\partial x_j} = O(|x|^{-2}), \quad |x|^2 = x_1^2 + x_2^2, \ \ j = 1, 2, \tag{9}$$

where o(.) and O(.) are Landau's notion.

The Neumann BVP problem. Find a regular solution $\mathbf{U}(\mathbf{u}, p_1, p_2)$ to systems (1) and (2) for $\mathbf{x} \in D$ satisfying the following boundary conditions:

$$\mathbf{P}\left(\frac{\partial}{\partial \mathbf{x}}, \mathbf{n}\right)\mathbf{U(z)} = \mathbf{f}(\mathbf{z}), \quad \frac{\partial}{\partial n} p_1(\mathbf{z}) = f_3(\mathbf{z}), \quad \frac{\partial}{\partial n} p_2(\mathbf{z}) = f_4(\mathbf{z}), \quad \mathbf{z} \in S, \tag{10}$$

where $\mathbf{f}(\mathbf{z})$, and $f_j(\mathbf{z})$, $j = 3, 4$, are known functions, $\mathbf{n}(\mathbf{z})$ is the external unit normal vector on S at $\mathbf{z}$ and $\mathbf{P}\left(\frac{\partial}{\partial \mathbf{x}}, \mathbf{n}\right)\mathbf{U}$ is the stress vector in the considered theory

$$\mathbf{P}\left(\frac{\partial}{\partial \mathbf{x}}, \mathbf{n}\right)\mathbf{U} = \mathbf{T}\left(\frac{\partial}{\partial \mathbf{x}}, \mathbf{n}\right)\mathbf{u} - \mathbf{n}(\beta_1 p_1 + \beta_2 p_2), \tag{11}$$

$\mathbf{T}\left(\frac{\partial}{\partial \mathbf{x}}, \mathbf{n}\right)\mathbf{u}$ is the stress vector in the classical theory of elasticity,

$$\mathbf{T}\left(\frac{\partial}{\partial \mathbf{x}}, \mathbf{n}\right)\mathbf{u}(\mathbf{x}) = \mu \frac{\partial}{\partial \mathbf{n}}\mathbf{u}(\mathbf{x}) + \lambda \mathbf{n} div \mathbf{u}(\mathbf{x}) + \mu \sum_{i=1}^{2} n_i(\mathbf{x}) grad u_i(\mathbf{x}).$$

Vector $\mathbf{U}(\mathbf{x})$ additionally has to satisfy the following decay conditions at infinity

$$\mathbf{U}(\mathbf{x}) = O(1), \quad \frac{\partial \mathbf{U}(\mathbf{x})}{\partial x_j} = O(|x|^{-2}), \quad |x|^2 = x_1^2 + x_2^2, \ \ j = 1, 2. \tag{12}$$

The uniqueness theorems

For a regular solutions of the Dirichlet and the Neumann BVPs in D Green's formulas:

$$\int_D [E(\mathbf{u},\mathbf{u}) - (\beta_1 p_1 + \beta_2 p_2) div\mathbf{u}] d\mathbf{x} = -\int_S \mathbf{u}\mathbf{P}(\partial\mathbf{y},\mathbf{n})\mathbf{U} d_y S,$$

$$\int_D \{\gamma(p_1-p_2)^2 + (k_{12}+k_{21}) gradp_1 gradp_2\} d\mathbf{x} \tag{13}$$

$$+\int_D \left\{k_1 (gradp_1)^2 + k_2 (gradp_2)^2\right\} d\mathbf{x} = -\int_S \mathbf{p}\mathbf{P}^{(1)}(\partial\mathbf{y},\mathbf{n})\mathbf{p} d_y S,$$

are valid, where

$$E(\mathbf{u},\mathbf{u}) = (\lambda+\mu)(divu)^2 + \mu\left(\frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2}\right)^2 + \mu\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right)^2.$$

$$\mathbf{P}^{(1)}(\partial\mathbf{x},\mathbf{n})\mathbf{p} = \begin{pmatrix} k_1 & k_{12} \\ k_{21} & k_2 \end{pmatrix} \frac{\partial\mathbf{p}}{\partial\mathbf{n}}, \qquad \mathbf{p} = (p_1, p_2).$$

For positive definiteness of the potential energy the inequalities $\mu > 0, \quad \lambda+\mu > 0$ are necessary and sufficient.

Now let us prove the following theorems.

Theorem 1. *The Dirichlet boundary value problem has at most one regular solution in the infinite domain D.*

Proof: Let the first BVP have in the domain D two regular solutions $\mathbf{U}^{(1)}$ and $\mathbf{U}^{(2)}$. Denote $\mathbf{U} = \mathbf{U}^{(1)} - \mathbf{U}^{(2)}$. The vectors $\mathbf{U}^{(1)}$ and $\mathbf{U}^{(2)}$ in the domain D must satisfy the condition (9); In this case formula (13) is valid and $\mathbf{U}(\mathbf{x}) = C, \quad \mathbf{x} \in D$, where C is a constant vector. But $\mathbf{U}$ on the boundary satisfies the condition $\mathbf{U} = 0$, which implies that $C = 0$ and $\mathbf{U}(\mathbf{x}) = 0, \ \mathbf{x} \in D$.

Theorem 2. *The regular solution of the Neumann boundary value problem $\boldsymbol{U} =$ const in the infinite domain D.*

Proof: For the exterior second homogeneous boundary value problem the vector $\mathbf{U}$ must satisfy condition at infinite (12). In this case, the formulas (13) are valid for a regular $\mathbf{U}$. Using these formulas, we obtain

$$u_1 = c_1 - \varepsilon x_2, \quad u_2 = c_2 + \varepsilon x_1, \quad p_1 = p_2 = const, \quad \mathbf{x} \in D,$$

where c_1, c_2, ε are constants. Bearing in mind (12), we have $\varepsilon = 0$, and

$$u_1 = c_1, \ \ u_2 = c_2, \ \ p_1 = p_2 = const, \quad \mathbf{x} \in D.$$

Explicit solution of the Dirichlet BVP for a plane with circular hole

A solution of system (2) with boundary conditions $p_1(\mathbf{z}) = f_3(\mathbf{z}),\quad p_2(\mathbf{z}) = f_4(\mathbf{z}),\quad \mathbf{z} \in S$ is sought in the form (5), where the functions φ and φ_1 are unknown in D. On the basis of boundary conditions we reformulate the problem in question as follows

$$\varphi(\mathbf{z}) = h(\mathbf{z}), \quad \varphi_1(\mathbf{z}) = h_1(\mathbf{z}), \quad \mathbf{z} \in S, \tag{14}$$

where

$$\begin{aligned} h &= \frac{1}{k_0}[(k_1 + k_{21})f_3 + (k_2 + k_{12})f_4], \\ h_1 &= \frac{1}{k_0}(k_1 + k_{21})(f_4 - f_3). \end{aligned} \tag{15}$$

Obviously the function φ is solution of the equation $\Delta\varphi = 0$ and it is represented in the form of the following series ([19], p. 281)

$$\varphi(\mathbf{x}) = \sum_{k=0}^{\infty} \left(\frac{R}{\rho}\right)^k (\mathbf{Y}_k \cdot \boldsymbol{\nu}_k(\psi)), \tag{16}$$

where

$$\mathbf{x}(x_1, x_2) = (\rho, \psi), \quad \rho^2 = x_1^2 + x_2^2, \quad \mathbf{Y}_k = (A_k, B_k),$$

$$\boldsymbol{\nu}_k = (\cos k\psi, \sin k\psi), \quad \mathbf{Y}_0 = (A_0, 0), \quad A_0 = \frac{1}{2\pi}\int_0^{2\pi} h(\theta)d\theta,$$

$$A_k = \frac{1}{\pi}\int_0^{2\pi} h(\theta)\cos k\theta d\theta, \quad B_k = \frac{1}{\pi}\int_0^{2\pi} h(\theta)\sin k\theta d\theta.$$

The regular metaharmonic function φ_1 in the domain D can be written as follows ([18], p. 99)

$$\varphi_1(\mathbf{x}) = \sum_{k=0}^{\infty} K_k(\lambda_0\rho)(\mathbf{Z}_k \cdot \boldsymbol{\nu}_k(\psi)), \tag{17}$$

where $K_k(\lambda_0\rho)$ is a modified Hankel's function of an imaginary argument, with the index k.

$K_k(\lambda_0\rho) \to 0,\quad \rho \to \infty;\quad \boldsymbol{\nu}_k = (\cos k\psi, \sin k\psi);\quad \mathbf{Z}_k = (C_k, D_k);\ \mathbf{Z}_0 = (C_0, 0);$ $C_0,\ C_k,\ D_k$ are the unknown quantities.

The function $h_1(z)$ in (15) can be represented in a Fourier series. Keeping in mind (17) and boundary conditions (14) we obtain the values of C_k and D_k

$$C_0 = \frac{1}{2\pi K_0(\lambda_0 R)}\int_0^{2\pi} h_1(\theta)d\theta, \qquad C_k = \frac{1}{\pi K_k(\lambda_0 R)}\int_0^{2\pi} h_1(\theta)\cos k\theta d\theta, \tag{18}$$

$$D_k = \frac{1}{\pi K_k(\lambda_0 R)} \int_0^{2\pi} h_1(\theta) \sin k\theta d\theta.$$

If we substitute the values of φ and φ_1 into (4), we find the functions $p_1(\mathbf{x})$ and $p_2(\mathbf{x})$ in D.

A solution $\mathbf{v}(\mathbf{x}) = (v_1, v_2)$ of homogeneous equation (6) is sought in the form [14]

$$\begin{aligned} v_1(\mathbf{x}) &= \frac{\partial}{\partial x_1}[\Phi_1 + \Phi_2] - \frac{\partial \Phi_3}{\partial x_2}, \\ v_2(\mathbf{x}) &= \frac{\partial}{\partial x_2}[\Phi_1 + \Phi_2] + \frac{\partial \Phi_3}{\partial x_1}, \end{aligned} \tag{19}$$

where Φ_1, Φ_2 and Φ_3 are scalar functions,

$$\begin{aligned} &\Delta\Phi_1 = 0, \quad \Delta\Delta\Phi_2 = 0, \quad \Delta\Delta\Phi_3 = 0, \\ &(\lambda + 2\mu)\frac{\partial}{\partial x_1}\Delta\Phi_2 - \mu\frac{\partial}{\partial x_2}\Delta\Phi_3 = 0, \\ &(\lambda + 2\mu)\frac{\partial}{\partial x_2}\Delta\Phi_2 + \mu\frac{\partial}{\partial x_1}\Delta\Phi_3 = 0. \end{aligned} \tag{20}$$

Taking into account (5) and boundary conditions (8), we can write

$$\mathbf{v}(\mathbf{z}) = \mathbf{\Psi}(\mathbf{z}), \tag{21}$$

where $\mathbf{\Psi}(\mathbf{z}) = \mathbf{f}(\mathbf{z}) - \mathbf{v}_0(\mathbf{z})$ is the known vector; $\varphi(z)$ and $\varphi_1(z)$ are defined by equalities (14). On the basis of equation $\Delta\varphi_0 = \varphi$ the function φ_0 is represented in the following form

$$\varphi_0(x) = \frac{R^2}{4}\sum_{k=2}^{\infty} \frac{1}{1-k}\left(\frac{R}{\rho}\right)^{k-2} (\mathbf{Y}_k \cdot \boldsymbol{\nu}_k(\psi)), \tag{22}$$

where $\mathbf{Y}_k$ is defined by (16).

In view of (20) we can represent the harmonic function Φ_1, biharmonic functions Φ_2 and Φ_3 in the form

$$\begin{aligned} \Phi_1 &= \sum_{k=0}^{\infty}\left(\frac{R}{\rho}\right)^{k} (\mathbf{X}_{k1} \cdot \boldsymbol{\nu}_k(\psi)), \\ \Phi_2 &= \sum_{k=0}^{\infty} R^2 \left(\frac{R}{\rho}\right)^{k-2} (\mathbf{X}_{k2} \cdot \boldsymbol{\nu}_k(\psi)), \\ \Phi_3 &= \frac{R^2(\lambda + 2\mu)}{\mu}\sum_{k=0}^{\infty}\left(\frac{R}{\rho}\right)^{k-2} (\mathbf{X}_{k2} \cdot \mathbf{s}_k(\psi)), \end{aligned} \tag{23}$$

where $\mathbf{X}_{ki} = (X_{ki1}, X_{ki2})$, $k = 1, 2$ are the unknown two-component vectors, $\boldsymbol{\nu}_k = (\cos k\psi, \sin k\psi)$, $\mathbf{s}_k = (-\sin k\psi, \cos k\psi)$. Using the formulas

$$\frac{\partial}{\partial x_1} = n_1 \frac{\partial}{\partial \rho} - \frac{n_2}{\rho}\frac{\partial}{\partial \psi}, \quad \frac{\partial}{\partial x_2} = n_2 \frac{\partial}{\partial \rho} + \frac{n_1}{\rho}\frac{\partial}{\partial \psi}$$

the boundary conditions (21) are rewritten in the form

$$v_n(\mathbf{z}) = \Psi_n(\mathbf{z}), \quad v_s(\mathbf{z}) = \Psi_s(\mathbf{z}), \quad \mathbf{z} \in S, \tag{24}$$

where v_n and $\Psi_n(\mathbf{z})$ are the normal components of the vectors $\boldsymbol{v} = (v_1, v_2)$ and $\Psi = (\Psi_1, \Psi_2)$ respectively; v_s and $\Psi_s(\mathbf{z})$ are the tangent components of the vectors $\boldsymbol{v} = (v_1, v_2)$ and $\Psi = (\Psi_1, \Psi_2)$ respectively. Substituting the equalities (19),(23) into (24), we get

$$\begin{aligned}
&v_n = \frac{\partial}{\partial \rho}(\Phi_1 + \Phi_2) - \frac{1}{\rho}\frac{\partial}{\partial \psi}\Phi_3,\\
&v_s = \frac{1}{\rho}\frac{\partial}{\partial \psi}(\Phi_1 + \Phi_2) + \frac{\partial}{\partial \rho}\Phi_3,\\
&\Psi_n = n_1\Psi_1 + n_2\Psi_2, \quad \Psi_s = -n_2\Psi_1 + n_1\Psi_2,\\
&\mathbf{n} = (n_1, n_2), \quad \mathbf{s} = (-n_2, n_1), \quad n_1 = \frac{x_1}{\rho}, \quad n_2 = \frac{x_2}{\rho}.
\end{aligned}\tag{25}$$

Let us expand the functions Ψ_n *and* Ψ_s in Fourier series, that Fourier coefficients are γ_k and δ_k :

$$\begin{aligned}
&\boldsymbol{\gamma}_0 = (\gamma_{01}, 0), \quad \boldsymbol{\gamma}_k = (\gamma_{k1}, \gamma_{k2}), \quad \boldsymbol{\delta}_0 = (\delta_{01}, 0), \quad \boldsymbol{\delta}_k = (\delta_{k1}, \delta_{k2}),\\
&\gamma_{01} = \frac{1}{\pi}\int_0^{2\pi} \Psi_n(\theta)d\theta, \quad \delta_{01} = \frac{1}{\pi}\int_0^{2\pi} \Psi_s(\theta)d\theta,\\
&\gamma_{k1} = \frac{1}{\pi}\int_0^{2\pi} \Psi_n(\theta)\cos k\theta d\theta, \quad \delta_{k1} = \frac{1}{\pi}\int_0^{2\pi} \Psi_s(\theta)\cos k\theta d\theta,\\
&\gamma_{k2} = \frac{1}{\pi}\int_0^{2\pi} \Psi_s(\theta)\sin k\theta d\theta, \quad \delta_{k2} = \frac{1}{\pi}\int_0^{2\pi} \Psi_n(\theta)\sin k\theta d\theta.
\end{aligned}\tag{26}$$

If we substitute (25) into (24), then obtained into (26), then passing to limit as $\rho \longrightarrow R$, for determining the unknown values we obtain the following system of algebraic equations whose solution is written in the following form:

$$X_{01i} = \frac{\gamma_{0i}R}{2}, \quad X_{k1i} = \frac{R(\gamma_{ki} + \delta_{ki})}{2k(\lambda + 3\mu)}[2\mu + (\lambda + \mu)k] - \frac{\gamma_{ki}R}{k},$$

$$X_{02i}=\frac{\delta_{0i}R\mu}{2},\quad X_{k2i}=\frac{(\gamma_{ki}+\delta_{ki})\mu}{2R(\lambda+3\mu)},\quad i=1,2,\quad k=1,2,....$$

Thus the solution of the Dirichlet boundary problem is represented by the sum (5) in which $\mathbf{v}(\mathbf{x})$ is defined by means of formula (19), $\mathbf{v}_0(\mathbf{x})$ by formula (7), $\varphi_0(\mathbf{x})$ by formula (22) and $\varphi_1(\mathbf{x})$ by formulas (17) and (18). It can be proved that if the functions $\mathbf{f}$ and $f_j,\quad j=3,4$ satisfy the following conditions on S

$$\mathbf{f}\in C^3(S),\quad f_j\in C^3(S),\quad j=3,4,$$

then the resulting series are absolutely and uniformly convergent.

Explicit solution of the Neumann BVP for a plane with circular hole

We sought the solution of the Neumann BVP in the form (4), where the functions φ and φ_1 are unknown in the domain D. Taking into account formulas (4), the boundary conditions can be rewritten as

$$\frac{\partial\varphi(\mathbf{z})}{\partial R}=h(\mathbf{z}),\qquad \frac{\partial\varphi_1(\mathbf{z})}{\partial R}=h_1(\mathbf{z}),\quad \mathbf{z}\in S. \tag{27}$$

$h(\mathbf{z})$ and $h_1(\mathbf{z})$ are given by (15), where $f_3=\dfrac{\partial p_1}{\partial R},\quad f_4=\dfrac{\partial p_2}{\partial R}$.

Thus for the unknown harmonic function φ we obtain the Neumann problem, the solution that is represented in the form of series ([19],p.282)

$$\varphi(\mathbf{x})=c_1-\sum_{k=1}^{\infty}\frac{R}{k}\left(\frac{R}{\rho}\right)^k(\mathbf{Y}_k\cdot\boldsymbol{\nu}_k(\psi)), \tag{28}$$

where c_1 is an arbitrary constant; $\mathbf{Y}_k=(A_k,B_k)$,

$$A_k=\frac{1}{\pi}\int_0^{2\pi}h(\theta)\cos k\theta d\theta,\quad B_k=\frac{1}{\pi}\int_0^{2\pi}h(\theta)\sin k\theta d\theta.$$

The metaharmonic function $\varphi_1(\mathbf{x})$ in the domain D can be written as (17), where $\mathbf{Z}_k=(C_k,D_k)$; $C_0,\ C_k,\ D_k$ are the unknown quantities. Keeping in mind (15) and boundary conditions (27), we obtain the values of $Z_0,\ C_k$ and D_k

$$C_0=\frac{1}{2\pi\lambda_0K_0'(\lambda_0R)}\int_0^{2\pi}h_1(\theta)d\theta,\qquad C_k=\frac{1}{\pi\lambda_0K_k'(\lambda_0R)}\int_0^{2\pi}h_1(\theta)\cos k\theta d\theta, \tag{29}$$

$$D_k=\frac{1}{\pi\lambda_0K_k'(\lambda_0R)}\int_0^{2\pi}h_1(\theta)\sin k\theta d\theta,$$

where

$$K_k'(\xi)=\frac{\partial K_k(\xi)}{\partial\xi},\quad \frac{\partial K_k(\lambda_0\rho)}{\partial\rho}=\lambda_0K_k'(\lambda_0\rho),\quad K_k'(\lambda_0R)\neq 0,\quad k=0,1,2,....$$

Taking into account (10) the boundary condition (9) for $\mathbf{T}\left(\frac{\partial}{\partial \mathbf{z}}, \mathbf{n}\right)\mathbf{v(z)}$ can be rewritten as

$$\mathbf{T}\left(\frac{\partial}{\partial \mathbf{z}}, \mathbf{n}\right)\mathbf{v(z)(z)} = \mathbf{\Omega(z)}, \quad \mathbf{z} \in S, \tag{30}$$

where

$$\mathbf{\Omega}(\ \mathbf{z}) = \mathbf{f(z)} + \mathbf{n(z)}[a\varphi_1(\mathbf{z}) + b\varphi(\mathbf{z})] - \mathbf{T}\left(\frac{\partial}{\partial \mathbf{z}}, \mathbf{n}\right)\mathbf{v}_0(\mathbf{z})$$

is the known vector, $\mathbf{\Omega} = (\Omega_1, \Omega_2)$; φ is defined by (28) and φ_1 - formulas (17) and (18); $\quad a = \beta_1 + \beta_2, \quad b = A_1\beta_1 + \beta_2$.

Let us rewrite the boundary conditions (30) in the form

$$\left[\mathbf{T}\left(\frac{\partial}{\partial \mathbf{z}}, \mathbf{n}\right)\mathbf{v(z)}\right]_n = \Omega_n(\mathbf{z}), \quad \left[\mathbf{T}\left(\frac{\partial}{\partial \mathbf{z}}, \mathbf{n}\right)\mathbf{v(z)}\right]_s = \Omega_s(\mathbf{z}), \tag{31}$$

where $\left[\mathbf{T}\left(\frac{\partial}{\partial \mathbf{z}}, \mathbf{n}\right)\mathbf{v(z)}\right]_n$ and $\Omega_n(\mathbf{z})$ are the normal components of the vectors $\mathbf{T}\left(\frac{\partial}{\partial \mathbf{z}}, \mathbf{n}\right)\mathbf{v}$ and $\mathbf{\Omega}(\mathbf{z})$ respectively; $\left[\mathbf{T}\left(\frac{\partial}{\partial \mathbf{z}}, \mathbf{n}\right)\mathbf{v(z)}\right]_s$ and $\Omega_s(\mathbf{z})$ are the tangent components of the vectors $\mathbf{T}\left(\frac{\partial}{\partial \mathbf{z}}, \mathbf{n}\right)\mathbf{v(z)})$ and $\mathbf{\Omega}(\mathbf{z})$ respectively.

$$\begin{aligned}
&\left[\mathbf{T}\left(\frac{\partial}{\partial \mathbf{z}}, \mathbf{n}\right)\mathbf{v(z)}\right]_n = (\lambda+\mu)\left[\frac{\partial v_n(\mathbf{z})}{\partial \rho}\right]_{\rho=R} + \frac{\lambda}{R}\frac{\partial v_s(\mathbf{z})}{\partial \psi},\\
&\left[\mathbf{T}\left(\frac{\partial}{\partial \mathbf{z}}, \mathbf{n}\right)\mathbf{v(z)}\right]_s = \mu\left[\frac{\partial v_s(\mathbf{z})}{\partial \rho}\right]_{\rho=R} + \frac{\mu}{R}\frac{\partial v_n(\mathbf{z})}{\partial \psi};\\
&\Omega_n(\mathbf{z}) = f_n(\mathbf{z}) + a\varphi_1(\mathbf{z}) + b\varphi(\mathbf{z}) - \left[\mathbf{T}\left(\frac{\partial}{\partial \mathbf{z}}, \mathbf{n}\right)\mathbf{v}_0(\mathbf{z})\right]_n,\\
&\Omega_s(\mathbf{z}) = f_s(\mathbf{z}) - \left[\mathbf{T}\left(\frac{\partial}{\partial \mathbf{z}}, \mathbf{n}\right)\mathbf{v}_0(\mathbf{z})\right]_s, \quad \mathbf{z} \in S.
\end{aligned} \tag{32}$$

v_n and v_s are defined from (25), $\mathbf{v}_0$ is defined by means of formula (7), where function $\varphi_0(x)$ is the solution of equation $\Delta\varphi_0 = \varphi$ and represented in the form [14]

$$\varphi_0(\mathbf{x}) = \frac{-R^3}{4}\sum_{k=2}^{\infty}\frac{1}{k(1-k)}\Big(\frac{R}{r}\Big)^{k-2}(\mathbf{Y}_k \cdot \boldsymbol{\nu}_k(\psi)),$$

Y_k are defined in (28); c_1 is an arbitrary constant.

Let us expand the functions Ω_n *and* Ω_s in Fourier series, those Fourier coefficients are $\gamma_k = (\gamma_{k1}, \gamma_{k2})$ and $\delta_k = (\delta_{k1}, \delta_{k2})$. Taking into account the formulas (25),(23) and (32), then passing to limit as $\rho \longrightarrow R$, for determining the unknown values we obtain the following system of algebraic equations

$$k[\lambda+2\mu(k+1)]X_{k1i}+$$

$$\left\{(\lambda+2\mu)(1-k)(2-k+\frac{\lambda+2\mu}{\mu}k)-\lambda kR^2\left[k+\frac{\lambda+2\mu}{\mu}(2-k)\right]\right\}X_{k2i}=\gamma_{ki}R^2,$$

$$-k(1+2k)X_{k1i}+R^2\left[k(3-2k)+\frac{\lambda+2\mu}{\mu}(k^2-3k+2)\right]X_{k2i}=\frac{\delta_{ki}R^2}{\mu},$$
$$i=1,2;\quad k=1,2,...,$$

where γ_{ki} and δ_{ki} are the Fourier coefficients of normal and tangential components of the vector $\mathbf{\Omega}(\mathbf{z})$ respectively.

We assume that the functions $\mathbf{f}$ and f_j, $(j=3,4)$ satisfies the following conditions on S

$$\mathbf{f}\in C^2(S),\quad f_j\in C^2(S),\quad j=3,4.$$

Under these conditions the resulting series are absolutely and uniformly convergent.

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