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ON THE EXISTENCE OF AN OPTIMAL ELEMENT IN QUASI-LINEAR NEUTRAL OPTIMAL PROBLEMS

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Abstract. For an optimal control problem involving neutral differential equation, whose right-hand side is linear with respect to prehistory of the phase velocity, existence theorems of optimal element are proved. Under element we imply the collection of delay parameters and initial functions, initial moment and vector, control and finally moment.

Keywords and phrases: Neutral differential equation, neutral optimal problem, optimal element, existence theorem.

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1. Formulation of main results

Let R^n_x be the n-dimensional vector space of points $x = (x^1, \dots, x^n)^T$, where T is the sign of transposition, let $a < t_{01} < t_{02} < t_{11} < t_{12} < b, 0 < \tau_1 < \tau_2, 0 < \sigma_1 < \sigma_2$ be given numbers with $t_{11} - t_{02} > \max\{\tau_2, \sigma_2\}$; suppose that $O \subset R^n_x$ is a open set and $U \subset R^r_u$ is a compact set, the function $F(t, x, y, u) = (f^0(t, x, y, u), f^1(t, x, y, u), ...,$ $f^n(t, x, y, u))^T$ is continuous on the set $I \times O^2 \times U$ and continuously differentiable with respect to x and y, where $I = [a, b]$; further, let Φ and Δ be sets of measurable initial functions $\varphi(t) \in K_0, t \in [\hat{\tau}, t_{02}]$ and $\varsigma(t) \in K_1, t \in [\hat{\tau}, t_{02}]$, respectively, where $\hat{\tau} = a - \max\{\tau_2, \sigma_2\}, K_0 \subset O$ is a compact set, $K_1 \subset R^n_x$ is a convex and compact set ; let Ω be a set of measurable control functions $u(t) \in U, t \in I$ and let $g^i(t_0, t_1, \tau, \eta, x_0, x_1), i = \overline{0, l}$ be continuous scalar functions on the set $[t_{01}, t_{02}] \times [t_{11}, t_{12}] \times [\tau_1, \tau_2] \times [\sigma_1, \sigma_2] \times X_0 \times O$, where $X_0 \subset O$ is a compact set.

To each element $w = (t_0, t_1, \tau, \sigma, x_0, \varphi, \varsigma, u) \in W = [t_{01}, t_{02}] \times [t_{11}, t_{12}] \times [\tau_1, \tau_2] \times [\sigma_1, \sigma_2] \times X_0 \times \Phi \times \Delta \times \Omega$ we assign the quasi-linear neutral differential equation

$$\dot{x}(t) = A(t)\dot{x}(t - \sigma) + f(t, x(t), x(t - \tau), u(t)), t \in [t_0, t_1] \qquad (1.1)$$

with the initial condition

$$x(t) = \varphi(t), \dot{x}(t) = \varsigma(t), t \in [\hat{\tau}, t_0), x(t_0) = x_0, \qquad (1.2)$$

where $A(t) = (a^i_j(t)), i, j = \overline{1, n}, t \in I$ is a given $n \times n$ -dimensional continuous matrix function, $f = (f^1, ..., f^n)^T$.

Remark 1.1. The symbol $\dot{x}(t)$ on the interval $[\hat{\tau}, t_0)$ is not connected with derivative of the function $\varphi(t)$.

Definition 1.1. Let $w = (t_0, t_1, \tau, \sigma, x_0, \varphi, \varsigma, u) \in W$. A function $x(t) = x(t; w) \in O, t \in [\hat{\tau}, t_1]$, is called a solution corresponding to the element w, if it satisfies condition (1.2) and is absolutely continuous on the interval $[t_0, t_1]$ and satisfies equation (1.1) almost everywhere (a.e.) on $[t_0, t_1]$.

Definition 1.2. An element $w = (t_0, t_1, \tau, \sigma, x_0, \varphi, \varsigma, u) \in W$ is said to be admissible if there exists the corresponding solution $x(t) = x(t; w)$ satisfying the condition

$$g(t_0, t_1, \tau, \sigma, x_0, x(t_1)) = 0, \tag{1.3}$$

where $g = (g^1, ..., g^l)$.

We denote the set of admissible elements by W_0. Now we consider the functional

$$J(w) = g^0(t_0, t_1, \tau, \sigma, x_0, x(t_1))+$$

$$\int_{t_0}^{t_1} \Big[a_0(t)\dot{x}(t-\sigma) + f^0(t, x(t), x(t-\tau), u(t))\Big]dt, w \in W_0,$$

where $x(t) = x(t; w)$, and $a_0(t) = (a_0^1(t), ..., a_0^n(t)), t \in I$ is a given continuous function.

Definition 1.3. An element $w_0 = (t_{00}, t_{10}, \tau_0, \sigma_0, x_{00}, \varphi_0, \varsigma_0, u_0) \in W_0$ is said to be optimal if

$$J(w_0) = \inf_{w \in W_0} J(w). \tag{1.4}$$

The problem (1.1)-(1.4) is called the quasi-linear neutral optimal problem.

Theorem 1.1. *There exists an optimal element w_0 if the following conditions hold:*

1.1. $W_0 \neq \emptyset$;

1.2. There exists a compact set $K_2 \subset O$ such that for an arbitrary $w \in W_0$

$$x(t; w) \in K_2, t \in [\hat{\tau}, t_1];$$

1.3. The sets

$$P(t,x) = \big\{F(t,x,y,u) : (y,u) \in K_0 \times U\big\}, (t,x) \in I \times O$$

and

$$P_1(t,x,y) = \big\{F(t,x,y,u) : u \in U\big\}, (t,x,y) \in I \times O^2$$

are convex.

Remark 1.2. Let K_0 and U be convex sets, and

$$F(t,x,y,u) = B(t,x)y + C(t,x)u.$$

Then the condition 1.3 of Theorem 1.1 holds.

Theorem 1.2. *There exists an optimal element w_0 if the conditions 1.1 and 1.2 of Theorem 1.1 hold, moreover the following conditions are fulfilled:*

1.4. The function $f(t,x,y,u)$ has a form

$$f(t,x,y,u) = D(t,x)y + E(t,x)u;$$

1.5. The sets K_0 and U are convex and for each fixed $(t,x) \in I \times O$ the function $f^0(t,x,y,u)$ is convex in $(y,u) \in K_0 \times U$.

The proof of existence of optimal delay parameters, initial functions and initial moment is the essential novelty in this work. Theorems of existence for optimal control problems involving various functional differential equations with fixed delay , initial function and moment are given in [1-5].

2. Auxiliary assertions

To each element $\mu = (t_0, \tau, \sigma, x_0, \varphi, \varsigma, u) \in \Pi = [t_{01}, t_{02}] \times [\tau_1, \tau_2] \times [\sigma_1, \sigma_2] \times O \times \Phi \times \Delta \times \Omega$ we will set in correspondence the functional differential equation

$$\dot{q}(t) = A(t)h(t_0, \varsigma, \dot{q})(t - \sigma) + f(t, q(t), h(t_0, \varphi, q)(t - \tau), u(t)) \tag{2.1}$$

with the initial condition

$$q(t_0) = x_0, \tag{2.2}$$

where the operator $h(t_0, \varphi, q)(t)$ is defined by the formula

$$h(t_0, \varphi, q)(t) = \begin{cases} \varphi(t), t \in [\hat{\tau}, t_0), \\ q(t), t \in [t_0, b]. \end{cases} \tag{2.3}$$

Definition 2.1. Let $\mu = (t_0, \tau, \sigma, x_0, \varphi, \varsigma, u) \in \Pi$. A function $q(t) = q(t; \mu) \in O, t \in [r_1, r_2]$, where $r_1 \in [t_{01}, t_{02}], r_2 \in [t_{11}, t_{12}]$, is called a solution corresponding to the element μ and defined on $[r_1, r_2]$, if $t_0 \in [r_1, r_2]$, and it satisfies condition (2.2) and is absolutely continuous on the interval $[r_1, r_2]$ and satisfies equation (2.1) a.e. on $[r_1, r_2]$.

Let $K_i \subset O, i = 3, 4$ be compact sets and K_4 contains a certain neighborhood of the set K_3.

Theorem 2.1. *Let $q_i(t) \in K_3, i = 1, 2, ...,$ be a solution corresponding to the element $\mu_i = (t_{0i}, \tau_i, \sigma_i, x_{0i}, \varphi_i, \varsigma_i, u_i) \in \Pi, i = 1, 2, ...,$ respectively,defined on the interval $[t_{0i}, t_{1i}]$, where $t_{1i} \in [t_{11}, t_{12}]$. Moreover,*

$$\lim_{i\to\infty} t_{0i} = t_{00}, \lim_{i\to\infty} \sigma_i = \sigma_0, \lim_{i\to\infty} t_{1i} = t_{10}. \tag{2.4}$$

Then there exist numbers $\delta > 0$ and $M > 0$ such that for a sufficiently large i_0 the solution $\psi_i(t)$ corresponding to the element $\mu_i, i \geq i_0$, respectively, is defined on the interval $[t_{00} - \delta, t_{10} + \delta] \subset I$. Moreover,

$$\psi_i(t) \in K_4, | \dot{\psi}_i(t) | \leq M, t \in [t_{00} - \delta, t_{10} + \delta]$$

and

$$\psi_i(t) = q_i(t), t \in [t_{0i}, t_{1i}] \subset [t_{00} - \delta, t_{10} + \delta].$$

Proof. Let $\varepsilon > 0$ be so small that a closed ε -neighborhood of the set $K_3 : K_3(\varepsilon) = \{x \in O : \exists \hat{x} \in K_3, |x - \hat{x}| \leq \varepsilon\}$ is contained $intK_4$. There exists a compact set $Q \subset R^n_x \times R^n_y$

$$K_3(\varepsilon) \times [K_0 \cup K_3(\varepsilon)] \subset Q \subset K_4 \times [K_0 \cup K_4]$$

and a continuously differentiable function $\chi : R^n_x \times R^n_y \to [0,1]$ such that

$$\chi(x,y) = \begin{cases} 1, (x,y) \in Q, \\ 0, (x,y) \notin K_4 \times [K_0 \cup K_4] \end{cases} \tag{2.5}$$

(see [6]). For each $i = 1, 2, ...$ the differential equation

$$\dot{\psi}(t) = A(t)h(t_{0i}, \varsigma_i, \dot{\psi})(t - \sigma_i) + \phi(t, \psi(t), h(t_{0i}, \varphi_i, \psi)(t - \tau_i), u_i(t)),$$

where

$$\phi(t, x, y, u) = \chi(x, y)f(t, x, y, u),$$

with the initial condition

$$\psi(t_{0i}) = x_{0i},$$

has the solution $\psi_i(t)$ defined on the interval I (see proof of Theorem 4.1,[7]). Since

$$(q_i(t), h(t_{0i}, \varphi_i, q_i)(t - \tau_i)) \in K_3 \times [K_0 \cup K_3] \subset Q, t \in [t_{0i}, t_{1i}],$$

(see (2.3)), therefore

$$\chi(q_i(t), h(t_{0i}, \varphi_i, q_i)(t - \tau_i)) = 1, t \in [t_{0i}, t_{1i}],$$

(see (2.5)),i.e.

$$\phi(t, q_i(t), h(t_{0i}, \varphi_i, q_i)(t - \tau_i), u_i(t)) = f(t, q_i(t), h(t_{0i}, \varphi_i, q_i)(t - \tau_i), u_i(t)),$$

$$t \in [t_{0i}, t_{1i}].$$

By the uniqueness

$$\psi_i(t) = q_i(t), t \in [t_{0i}, t_{1i}]. \tag{2.6}$$

There exists a number $M > 0$ such that

$$|\ \dot{\psi}_i(t)\ | \le M, t \in I, i = 1, 2, ... \tag{2.7}$$

Indeed, first of all we note that

$$|\ \phi(t, \psi_i(t), h(t_{0i}, \varphi_i, \psi_i)(t - \tau_i), u_i(t))\ | \le \sup\Big\{\ |\ \phi(t, x, y, u)\ | : t \in I, x \in K_4,$$

$$y \in K_4 \cup K_0, u \in U\Big\} := N_1, i = 1, 2, ...$$

It is not difficult to see that for sufficiently large i_0 we have

$$\Big[\frac{b - t_{0i}}{\sigma_i}\Big] = \Big[\frac{b - t_{00}}{\sigma_0}\Big] := d, i \ge i_0,$$

where $[\alpha]$ means the integer part of a number α, i.e.

$$t_{0i} + d\sigma_i \le b < t_{0i} + (d+1)\sigma_i.$$

If $t \in [a, t_{0i} + \sigma_i)$ then

$$| \dot{\psi}_i(t) | = | A(t)\varsigma_i(t-\sigma_i) + \phi(t, \psi_i(t), h(t_{0i}, \varphi_i, \psi_i)(t-\tau_i), u_i(t)) |$$

$$\leq \| A \| N_2 + N_1 := M_1,$$

where

$$\| A \| = \sup\Big\{ \ | A(t) |: t \in I \Big\}, N_2 = \sup\Big\{ \ | \xi |: \xi \in K_1 \Big\}.$$

Let $t \in [t_{0i} + \sigma_i, t_{0i} + 2\sigma_i)$ then

$$| \dot{\psi}_i(t) | \leq \| A \| | \dot{\psi}_i(t-\sigma_i) | + N_1 \leq \| A \| M_1 + N_1 := M_2$$

Continuing this process we obtain

$$| \dot{\psi}_i(t) | \leq \| A \| M_{j-1} + N_1 := M_j, t \in [t_{0i} + (j-1)\sigma_i, t_{0i} + j\sigma_i), j = 3, ..., d.$$

Moreover, if $t_{0i} + d\sigma_i < b$ then we have

$$| \dot{\psi}_i(t) | \leq M_{d+1}, t \in [t_{0i} + d\sigma_i, b].$$

It is clear that for $M = \max\{M_1, ..., M_{d+1}\}$ the condition (2.7) is fulfilled.

Further, there exists a number $\delta_0 > 0$ such that for an arbitrary $i = 1, 2..., [t_{0i} - \delta_0, t_{1i} + \delta_0] \subset I$ and the following conditions hold

$$|\psi_i(t_{0i}) - \psi_i(t)| \leq \int_t^{t_{0i}} \Big[|A(s) h(t_{0i}, \varsigma_i, \dot{\psi}_i)(s-\sigma_i)|$$

$$+ |\phi(s, \psi_i(s), h(t_{0i}, \varphi_i, \psi_i)(s-\tau_i), u_i(s))| ds \leq \varepsilon, t \in [t_{0i} - \delta_0, t_{0i} \Big],$$

$$|\psi_i(t) - \psi_i(t_{1i})| \leq \int_{t_{1i}}^{t} \Big[|A(s) h(t_{0i}, \xi_i, \dot{\psi}_i)(s-\sigma_i)|$$

$$+ |\phi(s, \psi_i(s), h(t_{0i}, \varphi_i, \psi_i)(s-\tau_i), u_i(s))| \Big] ds \leq \varepsilon, t \in [t_{1i}, t_{1i} + \delta_0].$$

These inequalities, taking into account $\psi_i(t_{0i}) \in K_3$ and $\psi_i(t_{1i}) \in K_3$, (see (2.6)), yield

$$(\psi_i(t), h(t_{0i}, \varphi_i, \psi_i)(t-\tau_i)) \in K_3(\varepsilon) \times [K_0 \cup K_3(\varepsilon)], t \in [t_{0i} - \delta_0, t_{1i} + \delta_0],$$

i.e.

$$\chi(\psi_i(t), h(t_{0i}, \varphi_i, \psi_i)(t-\tau_i)) = 1, t \in [t_{0i} - \delta_0, t_{1i} + \delta_0], i = 1, 2, ...,$$

Thus, $\psi_i(t)$ satisfies equation (2.1) and the conditions $\psi_i(t_{0i}) = x_{0i}, \psi_i(t) \in K_4, t \in [t_{0i} - \delta_0, t_{1i} + \delta_0]$, i.e. $\psi_i(t)$ is the solution corresponding to the element μ_i and defined on the interval $[t_{0i} - \delta_0, t_{1i} + \delta_0] \subset I$. Let $\delta \in (0, \delta_0)$, according to (2.4) for a sufficiently large i_0 we have

$$[t_{0i} - \delta_0, t_{1i} + \delta_0] \supset [t_{00} - \delta, t_{10} + \delta] \supset [t_{0i}, t_{1i}], i \geq i_0.$$

Consequently, $\psi_i(t), i \geq i_0$ solutions are defined on the interval $[t_{00}-\delta, t_{10}+\delta]$ and satisfy the conditions: $\psi_i(t) \in K_4, |\dot{\psi}_i(t)| \leq M, t \in [t_{00}-\delta, t_{10}+\delta]; \psi_i(t) = q_i(t), t \in [t_{0i}, t_{1i}]$, (see (2.6),(2.7)).

Theorem 2.2.([8]). *Let $p(t,u) \in R_p^m$ be a continuous function on the set $I \times U$ and let the set*

$$P(t) = \{p(t,u) : u \in U\}$$

be convex and

$$p_i(\cdot) \in L_1(I),\ p_i(t) \in P(t) \text{ a.e. on } I, i = 1,2,....$$

Moreover,

$$\lim_{i\to\infty} p_i(t) = p(t) \text{ weakly on } I.$$

Then

$$p(t) \in P(t) \text{ a.e. on } I$$

and there exists a measurable function $u(t) \in U, t \in I$ such that

$$p(t, u(t)) = p(t) \text{ a.e. on } I.$$

3. Proof of Theorem 1.1

Let

$$w_i = (t_{0i}, t_{1i}, \tau_i, \sigma_i, x_{0i}, \varphi_i, \varsigma_i, u_i) \in W_0, i = 1,2,...$$

be a minimizing sequence,i.e.

$$\lim_{i\to\infty} J(w_i) = \hat{J} = \inf_{w\in W_0} J(w).$$

Without loss of generality, we assume that

$$\lim_{i\to\infty} t_{0i} = t_{00}, \lim_{i\to\infty} t_{1i} = t_{10}, \lim_{i\to\infty} \tau_i = \tau_0, \lim_{i\to\infty} \sigma_i = \sigma_0, \lim_{i\to\infty} x_{0i} = x_{00}.$$

The set $\Delta \subset L_1([\hat{\tau}, t_{02}])$ is weakly compact (see Theorem 2.2), therefore we assume that

$$\lim_{i\to\infty} \varsigma_i(t) = \varsigma_0(t),\ \text{weakly in } t \in [\hat{\tau}, t_{02}]. \tag{3.1}$$

Introduce the following notation:

$$x_i^0(t) = \int_{t_{0i}}^{t} \Big[a_0(s)\dot{x}_i(s-\sigma_i) + f^0(s, x_i(s), x_i(s-\tau_i), u_i(s))\Big] ds,$$

$$x_i(t) = x(t; w_i), \rho_i(t) = (x_i^0(t), x_i(t))^T, t \in [t_{0i}, t_{1i}].$$

Obviously,

$$\begin{cases} \dot{\rho}_i(t) = \hat{A}(t)\dot{x}_i(t-\sigma_i) + F(t, x_i(t), x_i(t-\tau_i), u_i(t)), t \in [t_{0i}, t_{1i}], \\ x_i(t) = \varphi_i(t), t \in [\hat{\tau}, t_{0i}), \rho_i(t_{0i}) = (0, x_{0i})^T, \\ \dot{x}_i(t) = \varsigma_i(t), t \in [\hat{\tau}, t_{0i}), \end{cases}$$

where $\hat{A}(t)=(a_0(t)\ A(t))^T$. It is clear that

$$J(w_i)=g^0(t_{0i},t_{1i},\tau_i,\sigma_i,x_{0i},x_i(t_{1i}))+x_i^0(t_{1i}).$$

To each element $\mu=(t_0,\tau,\sigma,x_0,\varphi,\varsigma,u)\in\Pi$ we will set in correspondence the functional differential equation

$$\dot{z}(t)=\hat{A}(t)h(t_0,\varsigma,\dot{v})(t-\sigma)+F(t,v(t),h(t_0,\varphi,v)(t-\tau),u(t)),$$

with the initial condition

$$z(t_0)=z_0=(0,x_0)^T,$$

where $z(t)=(v^0(t),v(t))^T\in R_z^{1+n}$.

It is easy to see that

$$\begin{cases}\dot{\rho}_i(t)=\hat{A}(t)h(t_{0i},\varsigma_i,\dot{x}_i)(t-\sigma_i)+F(t,x_i(t),h(t_{0i},\varphi_i,x_i)(t-\tau_i),u_i(t)),t\in[t_{0i},t_{1i}],\\ \rho_i(t_{0i})=(0,x_{0i})^T\end{cases}$$

(see (2.3)). Thus, $\rho_i(t)$ is the solution corresponding to $\mu_i=(t_{0i},\tau_i,\sigma_i,x_{0i},\varphi_i,\varsigma_i,u_i)\in\Pi$ and defined on the interval $[t_{0i},t_{1i}]$. Since $x_i(t)\in K_2$, therefore in a similar way (see the proof of Theorem 2.1) we prove that $|\ \dot{x}_i(t)\ |\le N_3,t\in[t_{0i},t_{1i}],i=1,2,...,N_3>0$. Further, there exists a compact $H_1\subset H=\{z=(v^0,v)^T:v^0\in R_{v^0}^1,v\in O\}\subset R_z^{1+n}$ such that $\rho_i(t)\in H_1,t\in[t_{0i},t_{1i}]$.

Let $H_2\subset H$ be a compact set containing a certain neighborhood of the set H_1. By Theorem 2.1 there exists a number $\delta>0$ such that for a sufficiently large i_0 the solutions $z_i(t)=z(t;\mu_i),i\ge i_0$ are defined on the interval $[t_{00}-\delta,t_{10}+\delta]\subset I$ and the following conditions hold

$$\begin{cases}z_i(t)\in H_2,|\ \dot{z}_i(t)\ |\le M,t\in[t_{00}-\delta,t_{10}+\delta],\\ z_i(t)=\rho_i(t)=(x_i^0(t),x_i(t))^T,t\in[t_{0i},t_{1i}],i\ge i_0.\end{cases}\tag{3.2}$$

Thus, there exist numbers $N_4>0$ and $N_5>0$ such hat

$$\begin{cases}|F(t,v_i(t),h(t_{0i},\varphi_i,v_i)(t-\sigma_i),u_i(t))|\le N_5,\\ |\ h(t_{0i},\varsigma_i,\dot{v}_i)(t-\eta_i)\ |\le N_4,t\in[t_{00}-\delta,t_{10}+\delta],i\ge i_0.\end{cases}\tag{3.3}$$

The sequence $z_i(t)=(v_i^0(t),v_i(t))^T,t\in[t_{00}-\delta,t_{10}+\delta],i\ge i_0$ is uniformly bounded and equicontinuous. By the Arzela-Ascoli lemma, from this sequence we can extract a subsequence, which will again be denoted by $z_i(t),i\ge i_0$, that

$$\lim_{i\to\infty}z_i(t)=z_0(t)=(v_0^0(t),v_0(t))^T \text{ uniformly in } [t_{00}-\delta,t_{10}+\delta].$$

Further, from the sequence $\dot{z}_i(t),i\ge i_0$, we can extract a subsequence, which will again be denoted by $\dot{z}_i(t),i\ge i_0$, that

$$\lim_{i\to\infty}\dot{z}_i(t)=\gamma(t) \text{ weakly in } [t_{00}-\delta,t_{10}+\delta],$$

(see (3.2)). Obviously,

$$z_0(t) = \lim_{i\to\infty} z_i(t) = \lim_{i\to\infty}[z_i(t_{00}-\delta) + \int_{t_{00}-\delta}^{t} \dot z_i(s)ds]$$

$$= z_0(t_{00}-\delta) + \int_{t_{00}-\delta}^{t} \gamma(s)ds.$$

Thus, $\dot z_0(t) = \gamma(t)$ i.e.

$$\lim_{i\to\infty} \dot z_i(t) = \dot z_0(t) \text{ weakly in } [t_{00}-\delta, t_{10}+\delta].$$

Further, we have

$$z_i(t) = z_{0i} + \int_{t_{0i}}^{t} \Big[\hat A(s)h(t_{0i},\varsigma_i,\dot v_i)(s-\sigma_i) + F(s,v_i(s),h(t_{0i},\varphi_i,v_i)(s-\tau_i),u_i(s))\Big]ds$$

$$= z_{0i} + \vartheta_{1i}(t) + \vartheta_{2i} + \theta_{1i}(t) + \theta_{2i}, t\in[t_{00},t_{10}], i \ge i_0,$$

where

$$z_{0i} = (0,x_{0i})^T, \vartheta_{1i}(t) = \int_{t_{00}}^{t} \hat A(s)h(t_{0i},\varsigma_i,\dot v_i)(s-\sigma_i)ds,$$

$$\theta_{1i}(t) = \int_{t_{00}}^{t} F(s,v_i(s),h(t_{0i},\varphi_i,v_i)(s-\tau_i),u_i(s))ds,$$

$$\vartheta_{2i} = \int_{t_{0i}}^{t_{00}} \hat A(s)h(t_{0i},\varsigma_i,\dot v_i)(s-\sigma_i)ds,$$

$$\theta_{2i} = \int_{t_{0i}}^{t_{00}} F(s,v_i(s),h(t_{0i},\varphi_i,v_i)(s-\tau_i),u_i(s))ds.$$

Obviously, $\vartheta_{2i}\to 0$ and $\theta_{2i}\to 0$ as $i\to\infty$.

First of all we transform the expression $\vartheta_{1i}(t)$ for $t\in[t_{00},t_{10}]$. For this purpose, we consider two cases. Let $t\in[t_{00},t_{00}+\sigma_0]$, we have

$$\vartheta_{1i}(t) = \vartheta_{1i}^{(1)}(t) + \vartheta_{1i}^{(2)}(t),$$

where

$$\vartheta_{1i}^{(1)}(t) = \int_{t_{00}}^{t} \hat A(s)h(t_{00},\varsigma_i,\dot v_i)(s-\sigma_i)ds, \vartheta_{1i}^{(2)}(t) = \int_{t_{00}}^{t} \vartheta_{1i}^{(3)}(s)ds,$$

$$\vartheta_{1i}^{(3)}(s) = \hat A(s)\Big[h(t_{0i},\varsigma_i,\dot v_i)(s-\sigma_i) - h(t_{00},\varsigma_i,\dot v_i)(s-\sigma_i)\Big].$$

It is clear that

$$|\ \vartheta_{1i}^{(2)}(t)\ | \le \int_{t_{00}}^{t_{10}} |\ \vartheta_{1i}^{(3)}(s)\ | ds, t\in[t_{00},t_{10}]. \tag{3.4}$$

Suppose that $t_{0i}+\sigma_i > t_{00}$ for $i\ge i_0$. According to (2.3)

$$\vartheta_{1i}^{(3)}(s) = 0, s\in[t_{00},t_{0i}^{(1)})\cup(t_{0i}^{(2)},t_{1i}],$$

where

$$t_{0i}^{(1)}=\min\{t_{0i}+\sigma_i,t_{00}+\sigma_i\},t_{0i}^{(2)}=\max\{t_{0i}+\sigma_i,t_{00}+\sigma_i\}$$

Since

$$\lim_{i\to\infty}(t_{0i}^{(2)}-t_{0i}^{(1)})=0,$$

therefore,

$$\lim_{i\to\infty}\vartheta_{1i}^{(2)}(t)=0,\ \text{uniformly in}\ \ t\in[t_{00},t_{10}], \tag{3.5}$$

(see (3.3)). For $\vartheta_{1i}^{(1)}(t),t\in[t_{00},t_{00}+\sigma_0]$ we get

$$\vartheta_{1i}^{(1)}(t)=\int_{t_{00}-\sigma_i}^{t-\sigma_i}\hat{A}(s+\sigma_i)h(t_{00},\varsigma_i,\dot{v}_i)(s)ds=\vartheta_{1i}^{(4)}(t)+\vartheta_{1i}^{(5)}(t),$$

where

$$\vartheta_{1i}^{(4)}(t)=\int_{t_{00}-\sigma_0}^{t-\sigma_0}\hat{A}(s+\sigma_0)\varsigma_i(s)ds,\vartheta_{1i}^{(5)}(t)=\int_{t_{00}-\sigma_0}^{t-\sigma_0}\Big[\hat{A}(s+\sigma_i)-\hat{A}(s+\sigma_0)\Big]\varsigma_i(s)ds$$

$$+\int_{t_{00}-\sigma_i}^{t_{00}-\sigma_0}\hat{A}(s+\sigma_i)h(t_{00},\varsigma_i,\dot{v}_i)(s)ds+\int_{t-\sigma_0}^{t-\sigma_i}\hat{A}(s+\sigma_i)h(t_{00},\varsigma_i,\dot{v}_i)(s)ds.$$

Obviously,

$$\lim_{i\to\infty}\vartheta_{1i}^{(5)}(t)=0\ \text{uniformly in}\ \ t\in[t_{00},t_{00}+\sigma_0]$$

and

$$\lim_{i\to\infty}\vartheta_{1i}^{(1)}(t)=\lim_{i\to\infty}\vartheta_{1i}^{(4)}(t)=\int_{t_{00}-\sigma_0}^{t-\sigma_0}\hat{A}(s+\sigma_0)\varsigma_0(s)ds$$

$$=\int_{t_{00}}^{t}\hat{A}(s)\varsigma_0(s-\sigma_0)ds,t\in[t_{00},t_{00}+\sigma_0] \tag{3.6}$$

(see (3.1)).

Let $t\in[t_{00}+\sigma_0,t_{10}]$ then

$$\vartheta_{1i}^{(1)}(t)=\vartheta_{1i}^{(1)}(t_{00}+\sigma_0)+\vartheta_{1i}^{(6)}(t),$$

where

$$\vartheta_{1i}^{(6)}(t)=\int_{t_{00}+\sigma_0}^{t}\hat{A}(s)h(t_{0i},\varsigma_i,\dot{v}_i)(s-\sigma_i)ds.$$

Further,

$$\vartheta_{1i}^{(6)}(t)=\int_{t_{00}+\sigma_0}^{t}\hat{A}(s)h(t_{00},\varsigma_i,\dot{v}_i)(s-\sigma_i)ds+\int_{t_{00}+\sigma_0}^{t}\vartheta_{1i}^{(3)}(s)ds=\vartheta_{1i}^{(7)}(t)+\vartheta_{1i}^{(8)}(t).$$

It is clear that

$$\lim_{i\to\infty}\vartheta_{1i}^{(8)}(t)=0\ \text{uniformly in}\ \ t\in[t_{00}+\sigma_0,t_{10}],$$

(see (3.5)). For $\vartheta_{1i}^{(7)}(t), t\in[t_{00}+\sigma_0,t_{10}]$ we have

$$\vartheta_{1i}^{(7)}(t)=\int_{t_{00}+\sigma_0-\sigma_i}^{t-\sigma_i}\hat{A}(s+\sigma_i)h(t_{00},\varsigma_i,\dot{v}_i)(s)ds=\vartheta_{1i}^{(9)}(t)+\vartheta_{1i}^{(10)}(t),$$

where

$$\vartheta_{1i}^{(9)}(t)=\int_{t_{00}}^{t-\sigma_0}\hat{A}(s+\sigma_0)\dot{v}_i(s)ds, \vartheta_{1i}^{(10)}(t)=\int_{t_{00}+\sigma_0-\sigma_i}^{t_{00}}\hat{A}(s+\sigma_i)h(t_{00},\varsigma_i,\dot{v}_i)(s)ds$$

$$+\int_{t-\sigma_0}^{t-\sigma_i}\hat{A}(s+\sigma_i)h(t_{00},\varsigma_i,\dot{v}_i)(s)ds+\int_{t_{00}}^{t-\sigma_0}\Big[\hat{A}(s+\sigma_i)-\hat{A}(s+\sigma_0)\Big]\dot{v}_i(s)ds.$$

Obviously,

$$\lim_{i\to\infty}\vartheta_{1i}^{(10)}(t)=0 \text{ uniformly in } t\in[t_{00}+\sigma_0,t_{10}]$$

and

$$\lim_{i\to\infty}\vartheta_{1i}^{(1)}(t)=\lim_{i\to\infty}\vartheta_{1i}^{(1)}(t_{00}+\sigma_0)+\lim_{i\to\infty}\vartheta_{1i}^{(6)}(t)=\int_{t_{00}}^{t_{00}+\sigma_0}\hat{A}(t)\varsigma_0(t-\sigma_0)dt$$

$$+\lim_{i\to\infty}\vartheta_{1i}^{(9)}(t)=\int_{t_{00}}^{t_{00}+\sigma_0}\hat{A}(t)\varsigma_0(t-\sigma_0)dt+\int_{t_{00}}^{t-\sigma_0}\hat{A}(s+\sigma_0)\dot{v}_0(s)ds$$

$$=\int_{t_{00}}^{t_{00}+\sigma_0}\hat{A}(t)\varsigma_0(t-\sigma_0)dt+\int_{t_{00}+\sigma_0}^{t}\hat{A}(s)\dot{v}_0(s-\sigma_0)ds. \tag{3.7}$$

Now we transform the expression $\theta_{1i}(t)$ for $t\in[t_{00},t_{10}]$. We consider two cases again . Let $t\in[t_{00},t_{00}+\tau_0]$, we have

$$\theta_{1i}(t)=\theta_{1i}^{(1)}(t)+\theta_{1i}^{(2)}(t),$$

$$\theta_{1i}^{(1)}(t)=\int_{t_{00}}^{t}F(s,v_i(s),h(t_{00},\varphi_i,v_i)(s-\tau_i),u_i(s))ds, \theta_{1i}^{(2)}(t)=\int_{t_{00}}^{t}\theta_{1i}^{(3)}(s)ds,$$

$$\theta_{1i}^{(3)}(s)=F(s,v_i(s),h(t_{0i},\varphi_i,v_i)(s-\tau_i),u_i(s))-F(s,v_i(s),h(t_{00},\varphi_i,v_i)(s-\tau_i),u_i(s)).$$

It is clear that

$$\mid\theta_{1i}^{(2)}(t)\mid\leq\int_{t_{00}}^{t_{10}}\mid\theta_{1i}^{(3)}(s)\mid ds, t\in[t_{00},t_{10}]. \tag{3.8}$$

Suppose that $t_{0i}+\tau_i>t_{00}$ for $i\geq i_0$. According to (2.3)

$$\theta_{1i}^{(3)}(s)=0, s\in[t_{00},t_{0i}^{(3)})\cup(t_{0i}^{(4)},t_{1i}],$$

where

$$t_{1i}^{(3)}=\min\{t_{0i}+\tau_i,t_{00}+\tau_i\}, t_{1i}^{(4)}=\max\{t_{0i}+\tau_i,t_{00}+\tau_i\}.$$

Since

$$\lim_{i\to\infty}(t_{0i}^{(4)}-t_{0i}^{(3)})=0$$

therefore,

$$\lim_{i\to\infty} \theta_{1i}^{(2)}(t) = 0 \text{ uniformly in } \ t \in [t_{00}, t_{10}], \tag{3.9}$$

(see (3.3)). For $\theta_{1i}^{(1)}(t), t \in [t_{00}, t_{00} + \tau_0]$, we have

$$\theta_{1i}^{(1)}(t) = \int_{t_{00}-\tau_i}^{t-\tau_i} F(s+\tau_i, v_i(s+\tau_i), h(t_{00}, \varphi_i, v_i)(s), u_i(s+\tau_i))ds$$

$$= \theta_{1i}^{(4)}(t) + \theta_{1i}^{(5)}(t), i \geq i_0,$$

where

$$\theta_{1i}^{(4)}(t) = \int_{t_{00}-\tau_0}^{t-\tau_0} F(s+\tau_0, v_0(s+\tau_0), \varphi_i(s), u_i(s+\tau_i))ds,$$

$$\theta_{1i}^{(5)}(t) = \int_{t_{00}-\tau_i}^{t-\tau_i} F(s+\tau_i, v_i(s+\tau_i), h(t_{00}, \varphi_i, v_i)(s), u_i(s+\tau_i))ds$$

$$- \int_{t_{00}-\tau_0}^{t-\tau_0} F(s+\tau_0, v_0(s+\tau_0), \varphi_i(s), u_i(s+\tau_i))ds.$$

For $t \in [t_{00}, t_{00} + \tau_0]$ we obtain

$$\theta_{1i}^{(5)}(t) = \int_{t_{00}-\tau_i}^{t_{00}-\tau_0} F(s+\tau_i, v_i(s+\tau_i), h(t_{00}, \varphi_i, v_i)(s), u_i(s+\tau_i))ds$$

$$+ \int_{t_{00}-\tau_0}^{t-\tau_0} [F(s+\tau_i, v_i(s+\tau_i), \varphi_i(s), u_i(s+\tau_i)) - F(s+\tau_0, v_0(s+\tau_0), \varphi_i(s), u_i(s+\tau_i))]ds$$

$$+ \int_{t-\tau_0}^{t-\tau_i} F(s+\tau_i, v_i(s+\tau_i), h(t_{00}, \varphi_i, v_i)(s), u_i(s+\tau_i))ds.$$

Suppose that $\mid \tau_i - \tau_0 \mid \leq \delta$ as $i \geq i_0$. According to condition (3.3) and

$$\lim_{i\to\infty} F(s+\tau_i, v_i(s+\tau_i), y, u) = F(s+\tau_0, v_0(s+\tau_0), y, u)$$

uniformly in $(s, y, u) \in [t_{00} - \tau_0, t_{00}] \times K_0 \times U$, we have

$$\lim_{i\to\infty} \theta_{1i}^{(5)}(t) = 0 \text{ uniformly in } t \in [t_{00}, t_{00} + \tau_0].$$

From the sequence $F_i(s) = F(s+\tau_0, v_0(s+\tau_0), \varphi_i(s), u_i(s+\tau_i)), i \geq i_0, t \in [t_{00} - \tau_0, t_{00}]$, we extract a subsequence, which will again be denoted by $F_i(s), i \geq i_0$, such that

$$\lim_{i\to\infty} F_i(s) = F_0(s) \text{ weakly in the space } L_1([t_{00} - \tau_0, t_{00}]),$$

(see (3.3)). It is not difficult to see that

$$F_i(s) \in P(s+\tau_0, v_0(s+\tau_0)), s \in [t_{00} - \tau_0, t_{00}].$$

By Theorem 2.2

$$F_0(s) \in P(s+\tau_0, v_0(s+\tau_0)) \text{ a.e.} s \in [t_{00} - \tau_0, t_{00}]$$

and on the interval $[t_{00}-\tau_0, t_{00}]$ there exist measurable functions $\varphi_{01}(s) \in K_0, u_{01}(s) \in U$ such that

$$F_0(s) = F(s+\tau_0, v_0(s+\tau_0), \varphi_{01}(s), u_{01}(s)) \text{ a.e.} s \in [t_{00}-\tau_0, t_{00}].$$

Thus ,

$$\lim_{i\to\infty} \theta_{1i}^{(1)}(t) = \lim_{i\to\infty} \theta_{1i}^{(4)}(t) = \int_{t_{00}-\tau_0}^{t-\tau_0} F_0(s)ds$$

$$= \int_{t_{00}-\tau_0}^{t-\tau_0} F(s+\tau_0, v_0(s+\tau_0), \varphi_{01}(s), u_{01}(s))ds$$

$$= \int_{t_{00}}^{t} F(s, v_0(s), \varphi_{01}(s-\tau_0), u_{01}(s-\tau_0))ds, t \in [t_{00}, t_{00}+\tau_0]. \tag{3.10}$$

Let $t \in [t_{00}+\tau_0, t_{10}]$ then

$$\theta_{1i}^{(1)}(t) = \theta_{1i}^{(1)}(t_{00}+\tau_0) + \theta_{1i}^{(6)}(t), t \in [t_{00}+\tau_0, t_{10}],$$

where

$$\theta_{1i}^{(6)}(t) = \int_{t_{00}+\tau_0}^{t} F(s, v_i(s), h(t_{0i}, \varphi_i, v_i)(s-\tau_i), u_i(s))ds.$$

Further,

$$\theta_{1i}^{(6)}(t) = \theta_{1i}^{(7)}(t) + \theta_{1i}^{(8)}(t),$$

$$\theta_{1i}^{(7)}(t) = \int_{t_{00}+\tau_0}^{t} F(s, v_i(s), h(t_{00}, \varphi_i, v_i))(s-\tau_i), u_i(s))ds, \theta_{1i}^{(8)}(t) = \int_{t_{00}+\tau_0}^{t} \theta_{1i}^{(3)}(s)ds.$$

It is clear that

$$\lim_{i\to\infty} \theta_{1i}^{(8)}(t) = 0 \text{ uniformly in } \ t \in [t_{00}+\tau_0, t_{10}],$$

(see (3.8),(3.9)). For the expression $\theta_{1i}^{(7)}(t), t \in [t_{00}+\tau_0, t_{10}]$ we have

$$\theta_{1i}^{(7)}(t) = \int_{t_{00}+\tau_0-\tau_i}^{t-\tau_i} F(s+\tau_i, v_i(s+\tau_i), h(t_{00}, \varphi_i, v_i)(s), u_i(s+\tau_i))ds$$

$$= \theta_{1i}^{(9)}(t) + \theta_{1i}^{(10)}(t), i \geq i_0,$$

where

$$\theta_{1i}^{(9)}(t) = \int_{t_{00}}^{t-\tau_0} F(s+\tau_0, v_0(s+\tau_0), v_0(s), u_i(s+\tau_i))ds,$$

$$\theta_{1i}^{(10)}(t) = \int_{t_{00}+\tau_0-\tau_i}^{t-\tau_i} F(s+\tau_i, v_i(s+\tau_i), h(t_{00}, \varphi_i, v_i)(s), u_i(s+\tau_i))ds$$

$$- \int_{t_{00}}^{t-\tau_0} F(s+\tau_0, v_0(s+\tau_0), v_0(s), u_i(s+\tau_i))ds.$$

Clearly, for $t \in [t_{00}+\tau_0, t_{10}]$ we get

$$\theta_{1i}^{(10)}(t) = \int_{t_{00}+\tau_0-\tau_i}^{t_{00}} F(s+\tau_i, v_i(s+\tau_i), h(t_{00}, \varphi_i, v_i)(s), u_i(s+\tau_i))ds$$

$$+\int_{t_{00}}^{t-\tau_0}[F(s+\tau_i,v_i(s+\tau_i),v_i(s),u_i(s+\tau_i))-F(s+\tau_0,v_0(s+\tau_0),v_0(s),u_i(s+\tau_i))]ds$$

$$+\int_{t-\tau_0}^{t-\tau_i}F(s+\tau_i,v_i(s+\tau_i),h(t_{00},\varphi_i,v_i)(s),u_i(s+\tau_i))ds.$$

According to condition (3.3) and

$$\lim_{i\to\infty}F(s+\tau_i,v_i(s+\tau_i),v_i(s),u)=F(s+\tau_0,v_0(s+\tau_0),v_0(s),u)$$

uniformly in $(s,u)\in[t_{00},t_{10}-\tau_0]\times U$, we obtain

$$\theta_{1i}^{(10)}(t)=0 \text{ uniformly in } t\in[t_{00}+\tau_0,t_{10}].$$

From the sequence $F_i(s)=F(s+\tau_0,v_0(s+\tau_0),v_0(s),u_i(s+\tau_i)),i\ge i_0,t\in[t_{00},t_{10}-\tau_0]$, we extract a subsequence, which will again be denoted by $F_i(s),i\ge i_0$, such that

$$\lim_{i\to\infty}F_i(s)=F_0(s) \text{ weakly in the space } L_1([t_{00},t_{10}-\tau_0]).$$

It is not difficult to see that

$$F_i(s)\in P_1(s+\tau_0,v_0(s+\tau_0),v_0(s)),s\in[t_{00},t_{10}-\tau_0].$$

By Theorem 2.2

$$F_0(s)\in P_1(s+\tau_0,v_0(s+\tau_0),v_0(s)),a.e.s\in[t_{00},t_{10}-\tau_0]$$

and on the interval $[t_{00},t_{10}-\tau_0]$ there exists a measurable function $u_{02}(s)\in U$ such that

$$F_0(s)=F(s+\tau_0,v_0(s+\tau_0),v_0(s),u_{02}(s)) \text{ a.e.} s\in[t_{00},t_{10}-\tau_0].$$

Thus,

$$\lim_{i\to\infty}\theta_{1i}^{(1)}(t)=\lim_{i\to\infty}\theta_{1i}^{(1)}(t_{00}+\tau_0)+\lim_{i\to\infty}\theta_{1i}^{(9)}(t)=\int_{t_{00}}^{t_{00}+\tau_0}F(s,v_0(s),\varphi_{01}(s-\tau_0),u_{01}(s-\tau_0))ds$$

$$+\int_{t_{00}}^{t-\tau_0}F_0(s)ds=\int_{t_{00}}^{t_{00}+\tau_0}F(s,v_0(s),\varphi_{01}(s-\tau_0),u_{01}(s-\tau_0))ds$$

$$+\int_{t_{00}}^{t-\tau_0}F(s+\tau_0,v_0(s+\tau_0),v_0(s),u_{02}(s))ds=\int_{t_{00}}^{t_{00}+\tau_0}F(s,v_0(s),\varphi_{01}(s-\tau_0),u_{01}(s-\tau_0))ds$$

$$+\int_{t_{00}+\tau_0}^{t}F(s,v_0(s),v_0(s-\tau_0),u_{02}(s-\tau_0))ds,t\in[t_{00}+\tau_0,t_{10}], \tag{3.11}$$

(see (3.10)).

Introduce the following notation

$$\varphi_0(s)=\begin{cases}\hat{\varphi},s\in[\hat{\tau},t_{00}-\tau_0)\cup(t_{00},t_{02}],\\ \varphi_{01}(s),s\in[t_{00}-\tau_0,t_{00}],\end{cases}$$

$$u_0(s)=\begin{cases}\hat{u}, s\in[a,t_{00})\cup(t_{10},b],\\ u_{01}(s-\tau_0), s\in[t_{00},t_{00}+\tau_0],\\ u_{02}(s-\tau_0), s\in(t_{00}+\tau_0,t_{10}],\end{cases}$$

where $\hat{\varphi}\in K_0$ and $\hat{u}\in U$ are fixed points;

$$x_0(t)=\begin{cases}\varphi_0(t), t\in[\hat{\tau},t_{00}),\\ v_0(t), t\in[t_{00},t_{10}];\end{cases}$$

$$\dot{x}_0(t)=\varsigma_0(t), t\in[\hat{\tau},t_{00}),$$

(see Remark 1.1),

$$x_0^0(t)=v^0(t), t\in[t_{00},t_{10}].$$

Clearly, $w_0=(t_{00},t_{10},\tau_0,\sigma_0,x_{00},\varphi_0,\varsigma_0,u_0)\in W$. Taking into account (3.6),(3.7),(3.10) and (3.11) we obtain

$$x_0^0(t)=\int_{t_{00}}^t\Big[a_0(s)\dot{x}_0(s-\sigma_0)+f^0(s,x_0(s),x_0(s-\tau_0),u_0(s))\Big]ds, t\in[t_{00},t_{10}],$$

$$x_0(t)=x_{00}+\int_{t_{00}}^t\Big[A(s)\dot{x}_0(s-\sigma_0)+f(s,x_0(s),x_0(s-\tau_0),u_0(s))\Big]ds, t\in[t_{00},t_{10}].$$

It is not difficult to see that

$$\lim_{i\to\infty}(x_i^0(t_{1i}),x_i(t_{1i}))^T=\lim_{i\to\infty}\rho_i(t_{1i})=\lim_{i\to\infty}z_i(t_{1i}))$$

$$=\lim_{i\to\infty}[z_i(t_{1i})-z_i(t_{10})]+\lim_{i\to\infty}[z_i(t_{10})-z_0(t_{10})]+z_0(t_{10})=z_0(t_{10})$$

$$=(v^0(t_{10}),v_0(t_{10}))^T=(x_0^0(t_{10}),x_0(t_{10}))^T\in H,$$

(see (3.2)). Consequently,

$$0=\lim_{i\to\infty}g(t_{0i},t_{1i},\tau_i,\sigma_i,x_{0i},x_i(t_{1i}))=g(t_{00},t_{10},\tau_0,\sigma_0,x_{00},x_0(t_{10})),$$

i.e. the element w_0 is admissible and $x_0(t)=x(t;w_0), t\in[\hat{\tau},t_{10}]$.

Further, we have

$$\hat{J}=\lim_{i\to\infty}[g^0(t_{0i},t_{1i},\tau_i,\sigma_i,x_{0i},x_i(t_{1i}))+x_i^0(t_{1i})]=g(t_{00},t_{10},\tau_0,\sigma_0,x_{00},x_0(t_{10}))$$

$$+x_0^0(t_{10})=J(w_0).$$

Thus, w_0 is an optimal element.

4. Proof of Theorem 1.2

First of all we note that the sets $\Delta\subset L_1([\hat{\tau},t_{02}])$ and $\Omega\subset L_1(I)$ are weakly compacts (see Theorem 2.2). Let

$$w_i=(t_{0i},t_{1i},\tau_i,\sigma_i,x_{0i},\varphi_i,\varsigma_i,u_i)\in W_0, i=1,2,...$$

be a minimizing sequence,i.e.

$$\lim_{i\to\infty} J(w_i) = \hat{J} = \inf_{w\in W_0} J(w).$$

Without loss of generality, we assume that

$$\lim_{i\to\infty} t_{0i} = t_{00}, \lim_{i\to\infty} t_{1i} = t_{10}, \lim_{i\to\infty} \tau_i = \tau_0, \lim_{i\to\infty} \sigma_i = \sigma_0, \lim_{i\to\infty} x_{0i} = x_{00},$$

$$\begin{cases} \lim_{i\to\infty} \varphi_i(t) = \varphi_0(t), \text{weakly on } [\hat{\tau}, t_{02}], \\ \lim_{i\to\infty} \varsigma_i(t) = \varsigma_0(t), \text{weakly on } [\hat{\tau}, t_{02}], \\ \lim_{i\to\infty} u_i(t) = u_0(t) \text{ weakly on } I. \end{cases} \tag{4.1}$$

(see (3.1)).

To each element $\mu = (t_0, \tau, \sigma, x_0, \varphi, \varsigma, u) \in \Pi$ we will set in correspondence the functional differential equation

$$\dot{\zeta}(t) = A(t)h(t_0, \varsigma, \dot{\zeta})(t-\sigma) + C(t, \zeta(t))h(t_0, \varphi, \zeta)(t-\tau) + D(t, \zeta(t))u(t)$$

with the initial condition

$$\zeta(t_0) = x_0$$

It is easy to see that for $x_i(t) = x(t; w_i)$ we have

$$\begin{cases} \dot{x}_i(t) = A(t)h(t_0, \varsigma, \dot{x}_i)(t-\sigma_i) + C(t, x_i(t))h(t_{0i}, \varphi_i, x_i)(t-\tau_i) + \\ D(t, x_i(t))u_i(t), t \in [t_{0i}, t_{1i}], \\ x_i(t_{0i}) = x_{0i}. \end{cases}$$

Thus, $x_i(t) \in K_2$ is the solution corresponding to $\mu_i = (t_{0i}, \tau_i, \sigma_i, x_{0i}, \varphi_i, \varsigma_i, u_i)$ and defined on the interval $[t_{0i}, t_{1i}]$. Let $\hat{K}_2 \subset O$ be a compact set containing a certain neighborhood of the set K_2. By Theorem 2.1 there exists a number $\delta > 0$ such that for a sufficiently large i_0 the solutions $\zeta_i(t) = \zeta(t; \mu_i), i \geq i_0$ are defined on the interval $[t_{00} - \delta, t_{10} + \delta] \subset I$ and

$$\zeta_i(t) \in \hat{K}_2, t \in [t_{00} - \delta, t_{10} + \delta],\ \zeta_i(t) = x_i(t), t \in [t_{0i}, t_{1i}], i \geq i_0.$$

After this (see the proof of Theorem 1.1) we prove in the standard way that

$$\lim_{i\to\infty} \zeta_i(t) = \zeta_0(t) \text{ uniformly in } t \in [t_{00} - \delta, t_{10} + \delta],$$

and

$$\lim_{i\to\infty} \dot{\zeta}_i(t) = \dot{\zeta}_0(t) \text{ weakly on } t \in [t_{00} - \delta, t_{10} + \delta],$$

where $\zeta_0(t)$ is the solution corresponding to the element $\mu_0 = (t_{00}, \tau_0, \sigma_0, x_{00}, \varphi_0, \varsigma_0, u_0)$, defined on the interval $[t_{00} - \delta, t_{10} + \delta]$ and satisfying the condition $\zeta_0(t_{00}) = x_{00}$. Moreover,

$$\lim_{i\to\infty} x_i(t_{1i}) = \lim_{i\to\infty} \zeta_i(t_{1i}) = \lim_{i\to\infty} [\zeta_i(t_{1i}) - \zeta_i(t_{10})]$$

$$+\lim_{i\to\infty}[\zeta_i(t_{10})-\zeta_0(t_{10})]+\zeta_0(t_{10})=\zeta_0(t_{10}),$$

Hence,

$$0=\lim_{i\to\infty} g(t_{0i},t_{1i},\tau_i,\sigma_i,x_{0i},x_i(t_{1i}))=g(t_{00},t_{10},\tau_0,\sigma_0,x_{00},\zeta_0(t_{10})).$$

Introduce the following notation

$$x_0(t)=\begin{cases}\varphi_0(t), t\in[\hat{\tau},t_{00}),\\ \zeta_0(t), t\in[t_{00},t_{10}]\end{cases} \tag{4.2}$$

$$\dot{x}_0(t)=\varsigma_0(t), t\in[\hat{\tau},t_{00}), \tag{4.3}$$

(see Remark 1.1).

Clearly the function $x_0(t)$ is the solution corresponding to the element $w_0=(t_{00},t_{10},\tau_0,\sigma_0,x_{00},\varphi_0,\varsigma_0,u_0)\in W$ and satisfying the condition

$$g(t_{00},t_{10},\tau_0,\sigma_0,x_{00},x_0(t_{10}))=0,$$

i.e. $w_0\in W_0$.

Now we prove optimality of the element w_0. We have,

$$\lim_{i\to\infty} g^0(t_{0i},t_{1i},\tau_i,\sigma_i,x_{0i},x_i(t_{1i}))=g^0(t_{00},t_{10},\tau_0,\sigma_0,x_{00},x_0(t_{10})),$$

$$\int_{t_{0i}}^{t_{1i}} a_0(t)\dot{x}_i(t-\sigma_i)dt=\int_{t_{0i}}^{t_{1i}} a_0(t)h(t_{1i},\xi_i,\dot{\zeta}_i)(t-\sigma_i)dt,$$

$$\int_{t_{0i}}^{t_{1i}} f^0(t,x_i(t),x_i(t-\tau_i),u_i(t))dt=\int_{t_{0i}}^{t_{1i}} f^0(t,\zeta_i(t),h(t_{0i},\varphi_i,\zeta_i)(t-\tau_i),u_i(t))dt.$$

In a similar way (see proof of Theorem 1.1) it can be proved that

$$\int_{t_{0i}}^{t_{1i}} a_0(t)h(t_{1i},\varsigma_i,\dot{\zeta}_i)(t-\eta_i)dt=\varrho_{1i}+\varrho_{2i}+\varrho_{3i}$$

$$\int_{t_{0i}}^{t_{1i}} f^0(t,\zeta_i(t),h(t_{0i},\varphi_i,\zeta_i)(t-\tau_i),u_i(t))dt=\gamma_{1i}+\gamma_{2i}+\gamma_{3i},$$

where

$$\varrho_{1i}=\int_{t_{00}-\sigma_0}^{t_{00}} a_0(t+\sigma_0)\xi_i(t)dt, \varrho_{2i}=\int_{t_{00}}^{t_{10}-\sigma_0} a_0(t+\sigma_0)\dot{v}_i(t)dt$$

$$\gamma_{1i}=\int_{t_{00}-\tau_0}^{t_{00}} f^0(t+\tau_0,\zeta_0(t+\tau_0),\varphi_i(t),u_i(t+\tau_i))dt,$$

$$\gamma_{2i}=\int_{t_{00}}^{t_{10}-\tau_0} f^0(t+\tau_0,\zeta_0(t+\tau_0),\zeta_0(t),u_i(t+\tau_i))dt$$

and

$$\lim_{i\to\infty}\varrho_{3i}=0, \lim_{i\to\infty}\gamma_{3i}=0.$$

The functionals

$$\int_{t_{00}-\tau_0}^{t_{00}} f^0(t+\tau_0,\zeta_0(t+\tau_0),\varphi(t),u(t))dt,\ (\varphi,u)\in\Delta\times\Omega$$

and

$$\int_{t_{00}}^{t_{10}-\tau_0} f^0(t+\tau_0,\zeta_0(t+\tau_0),\zeta_0(t),u(t))dt, u\in\Omega$$

are lower semicontinuous (see [3]).

It is not difficult to see that, if

$$\lim_{i\to\infty} u_i(t)=u_0(t) \text{ weakly on } I$$

then

$$\lim_{i\to\infty} u_i(t+\tau_i)=u_0(t+\tau_0) \text{ weakly on } [t_{00}-\tau_0,t_{10}-\tau_0],$$

(see (4.1)). Using the latter and above given relations, we get

$$\hat{J}=\lim_{i\to\infty} J(w_i)=\lim_{i\to\infty}[g^0(t_{0i},t_{1i},\tau_i,\sigma_i,x_{0i},x_i(t_{1i}))+\varrho_{1i}+\varrho_{2i}+\varrho_{3i}$$

$$+\gamma_{1i}+\gamma_{2i}+\gamma_{3i}]=g^0(t_{00},t_{10},\tau_0,\sigma_0,x_{00},x_0(t_{10}))+\lim_{i\to\infty}[\varrho_{1i}+\varrho_{2i}]$$

$$+\lim_{i\to\infty}[\gamma_{1i}+\gamma_{2i}]\geq g^0(t_{00},t_{10},\tau_0,\sigma_0,x_{00},x_0(t_{10}))+\int_{t_{00}-\sigma_0}^{t_{00}} a_0(t+\sigma_0)\zeta_0(t)dt$$

$$+\int_{t_{00}}^{t_{10}-\sigma_0} a_0(t+\sigma_0)\dot{\zeta}_0(t)dt+\int_{t_{00}-\tau_0}^{t_{00}} f^0(t+\tau_0,\zeta_0(t+\tau_0),\varphi_0(t),u_0(t+\tau_0))dt$$

$$+\int_{t_{00}}^{t_{10}-\tau_0} f^0(t+\tau_0,\zeta_0(t+\tau_0),\zeta_0(t),u_0(t+\tau_0))dt=g^0(t_{00},t_{10},\tau_0,\sigma_0,x_{00},x_0(t_{10}))$$

$$+\int_{t_{00}}^{t_{10}}\Big[a_0(t)\dot{x}_0(t-\sigma_0)+f^0(t,x_0(t),x_0(t-\tau_0),u_0(t))\Big]dt=J(w_0),$$

(see (4.2),(4.3)). Here, by definition of $\hat{J}$, the inequality is impossible. The optimality of the element w_0 has been proved.

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