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VARIATION FORMULAS OF SOLUTION FOR A CLASS OF CONTROLLED NEUTRAL FUNCTIONAL-DIFFERENTIAL EQUATION CONSIDERING DELAY FUNCTION PERTURBATION AND THE CONTINUOUS INITIAL CONDITION

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Abstract. Variation formulas of solution are obtained for linear with respect to prehistory of the phase velocity (quasi-linear) controlled neutral functional-differential equation with variable delays. The effects of delay function perturbation and continuous initial condition are detected in the variation formulas.

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Let $I = [a, b]$ be a finite interval and let $\mathbb{R}^n$ be the n-dimensional vector space of points $x = (x^1, ..., x^n)^T$, where T is the sign of transposition. Suppose that $O \subset \mathbb{R}^n_x$ and $U_0 \subset \mathbb{R}^r_u$ are open sets. Let the n-dimensional function $f(t, x, y, u)$ satisfy the following conditions: for almost all $t \in I$, the function $f(t, \cdot) : O^2 \times U_0 \to \mathbb{R}^n_x$ is continuously differentiable; for any $(x, y, u) \in O^2 \times U_0$, the functions $f(t, x, y, u), f_x(\cdot), f_y(\cdot), f_u(\cdot)$ are measurable on I; for arbitrary compacts $K \subset O, U \subset U_0$ there exists a function $m_{K,U}(\cdot) \in L(I, [0, \infty))$, such that for any $(x, y, u) \in K^2 \times U$ and for almost all $t \in I$ the following inequality is fulfilled

$$| f(t, x, y, u) | + | f_x(\cdot) | + | f_y(\cdot) | + | f_u(\cdot) | \leq m_{K,U}(t).$$

Further, let D be the set of continuously differentiable scalar functions (delay functions) $\tau(t), t \in I$, satisfying the conditions:

$$\tau(t) < t, \dot{\tau}(t) > 0, \inf\{\tau(a) : \tau \in D\} := \hat{\tau} > -\infty.$$

Let Φ be the set of continuously differentiable initial functions $\varphi(t) \in O, t \in I_1 = [\hat{\tau}, b]$ and let $\Omega = \{u \in E_u : clu(I) \subset U_0\}$ be the set of control functions, where E_u is the space of bounded measurable functions $u : I \to \mathbb{R}^r_u$ and $u(I) = \{u(t) : t \in I\}$

To each element $\mu = (t_0, \tau, \varphi, u) \in \Lambda = [a, b) \times D \times \Omega$ we assign the quasi-linear controlled neutral functional-differential equation

$$\dot{x}(t) = A(t)\dot{x}(\sigma(t)) + f(t, x(t), x(\tau(t)), u(t)) \tag{1}$$

with the continuous initial condition

$$x(t) = \varphi(t), t \in [\hat{\tau}, t_0], \tag{2}$$

where $A(t)$ is a given continuous matrix function with dimension $n \times n; \sigma \in D$ is a fixed delay function.

Definition 1. Let $\mu=(t_0,\tau,\varphi,u)\in\Lambda$. A function $x(t)=x(t;\mu)\in O, t\in[\hat{\tau},t_1], t_1\in(t_0,b]$, is called a solution of equation (1) with the initial condition (2) or a solution corresponding to the element μ and defined on the interval $[\hat{\tau},t_1]$, if $x(t)$ satisfies condition (2) and is absolutely continuous on the interval $[t_0,t_1]$ and satisfies equation (1) almost everywhere on $[t_0,t_1]$.

Let $\mu_0=(t_{00},\tau_0,\varphi_0,u_0)\in\Lambda$ be a given element and let $x_0(t)$ be the solution corresponding to μ_0 and defined on $[\hat{\tau},t_{10}]$, with $a<t_{00}<t_{10}<b$.

Let us introduce the set of variations

$$V=\Big\{\delta\mu=(\delta t_0,\delta\tau,\delta\varphi,\delta u):|\,\delta t_0\,|\le\alpha,\|\,\delta\tau\,\|\le\alpha,$$

$$\delta\varphi=\sum_{i=1}^{k}\lambda_i\delta\varphi_i,|\,\lambda_i\,|\le\alpha,i=\overline{1,k},\|\,\delta u\,\|\le\alpha\Big\}.$$

Here

$$\delta t_0\in\mathbb{R},\delta\tau\in D-\tau_0,\|\delta\tau\|=\sup\{|\delta\tau(t)|:t\in I\},\delta u\in\Omega-u_0$$

and

$$\delta\varphi_i\in\Phi-\varphi_0,i=\overline{1,k}$$

are fixed functions, $\alpha>0$ is a fixed number.

There exist numbers $\delta_1>0$ and $\varepsilon_1>0$ such that for arbitrary $(\varepsilon,\delta\mu)\in(0,\varepsilon_1]\times V$ the element $\mu_0+\varepsilon\delta\mu\in\Lambda$ and there corresponds the solution $x(t;\mu_0+\varepsilon\delta\mu)$ defined on the interval $[\hat{\tau},t_{10}+\delta_1]\subset I_1$ ([1],Theorem 3).

Due to the uniqueness, the solution $x(t;\mu_0)$ is a continuation of the solution $x_0(t)$ on the interval $[\hat{\tau},t_{10}+\delta_1]$. Therefore, the solution $x_0(t)$ is assumed to be defined on the interval $[\hat{\tau},t_{10}+\delta_1]$.

Let us define the increment of the solution $x_0(t)=x(t;\mu_0)$:

$$\Delta x(t;\varepsilon\delta\mu)=x(t;\mu_0+\varepsilon\delta\mu)-x_0(t),\forall(t,\varepsilon,\delta\mu)\in[\hat{\tau},t_{10}+\delta_1]\times(0,\varepsilon_1]\times V.$$

Theorem 1. *Let the following conditions hold:*

1) *The function $f_0(z), z=(t,x,y)\in I\times O^2$ is bounded, where $f_0(t,x,y)=f(t,x,y,u_0(t))$;*

2) *There exists the limit*

$$\lim_{z\to z_0}f_0(z)=f_0^-,z\in(a,t_{00}]\times O^2$$

where $z_0=(t_{00},\varphi_0(t_{00}),\varphi_0(\tau_0(t_{00})))$. Then there exist numbers $\varepsilon_2\in(0,\varepsilon_1)$ and $\delta_2\in(0,\delta_1)$ such that

$$\Delta x(t;\varepsilon\delta\mu)=\varepsilon\delta x(t;\delta\mu)+o(t;\varepsilon\delta\mu) \tag{3}$$

for arbitrary $(t,\varepsilon,\delta\mu)\in[t_{00},t_{10}+\delta_2]\times(0,\varepsilon_2]\times V^-$, where $V^-=\{\delta\mu\in V:\delta t_0\le 0\}$ and

$$\delta x(t;\delta\mu)=Y(t_{00}-;t)[\dot{\varphi}_0(t_{00})-A(t_{00})\dot{\varphi}_0(\sigma(t_{00}))-f_0^-]\delta t_0+\beta(t;\delta\mu), \tag{4}$$

$$\beta(t;\delta\mu)=\Psi(t_{00};t)\delta\varphi(t_{00})+\int_{\tau_0(t_{00})}^{t_{00}}Y(\gamma_0(s);t)f_{0y}[\gamma_0(s)]\dot{\gamma}_0(s)\delta\varphi(s)ds$$

$$+\int_{\sigma(t_{00})}^{t_{00}} Y(\varrho(s);t)A(\varrho(s))\dot{\varrho}(s)\dot{\delta}\varphi(s)ds + \int_{t_{00}}^{t} Y(s;t)f_{0y}[s]\dot{x}_0(\tau_0(s))\delta\tau(s)ds$$

$$+\int_{t_{00}}^{t} Y(s;t)f_{0u}[s]\delta u(s)]ds; \qquad (5)$$

$$\lim_{\varepsilon\to 0}\frac{o(t;\varepsilon\delta\mu)}{\varepsilon} = 0 \ uniformly\ for\ \ (t,\delta\mu)\in[t_{00},t_{10}+\delta_2]\times V^-,$$

$Y(s;t)$ and $\Psi(s;t)$ are $n\times n$-matrix functions satisfying the system

$$\begin{cases}\Psi_s(s;t) = -Y(s;t)f_{0x}[t] - Y(\gamma_0(s);t)f_{0y}[\gamma_0(s)]\dot{\gamma}_0(s),\\ Y(s;t) = \Psi(s;t) + Y(\varrho(s);t)A(\varrho(s))\dot{\varrho}(s), s\in[t_{00}-\delta_2,t]\end{cases}$$

and the condition

$$\Psi(s;t) = Y(s;t) = \begin{cases}H, s=t,\\ \Theta, s>t;\end{cases}$$

$$f_{0x}[s] = f_{0x}(s,x_0(s),x_0(\tau_0(s)));$$

$\gamma_0(s)$ is the inverse function of $\tau_0(t)$, $\varrho(s)$ is the inverse function of $\sigma(t)$, H is the identity matrix and Θ is the zero matrix.

Some comments. The function $\delta x(t;\delta\mu)$ is called the variation of the solution $x_0(t), t\in[t_{00},t_{10}+\delta_2]$, and the expression (4) is called the variation formula.

The addend

$$\int_{t_{00}}^{t} Y(s;t)f_{0y}[s]\dot{x}_0(\tau_0(s))\delta\tau(s)ds$$

in formula (5) is the effect of perturbation of the delay function $\tau_0(t)$.

The expression

$$Y(t_{00}-;t)[\dot{\varphi}_0(t_{00}) - A(t_{00})\dot{\varphi}_0(\sigma(t_{00})) - f_0^-]\delta t_0$$

is the effect of continuous initial condition (2) and perturbation of the initial moment t_{00}.

The expression

$$\Psi(t_{00};t)\delta\varphi(t_{00}) + \int_{\tau_0(t_{00})}^{t_{00}} Y(\gamma_0(s);t)f_{0y}[\gamma_0(s)]\dot{\gamma}_0(s)\delta\varphi(s)ds$$

$$+\int_{\sigma(t_{00})}^{t_{00}} Y(\varrho(s);t)A(\varrho(s))\dot{\varrho}(s)\dot{\delta}\varphi(s)ds$$

in formula (5) is the effect of perturbation of the initial function $\varphi_0(t)$.

The expression

$$\int_{t_{00}}^{t} Y(s;t)f_{0u}[s]\delta u(s)]ds$$

in formula (5) is the effect of perturbation of the control function $u_0(t)$.

Variation formulas of solution for various classes of neutral functional differential equations without perturbation of delay are given in [2-4]. The variation formula of solution plays the basic role in proving the necessary conditions of optimality and under sensitivity analysis of mathematical models [5-8]. Finally we note that the variation formula allows to obtain an approximate solution of the perturbed equation

$$\dot{x}(t) = A(t)\dot{x}(\sigma(t)) + f(t, x(t), x(\tau_0(t) + \varepsilon\delta\tau(t)), u_0(t) + \varepsilon\delta u(t))$$

with the perturbed initial condition

$$x(t) = \varphi_0(t) + \varepsilon\delta\varphi(t), t \in [\hat{\tau}, t_{00} + \varepsilon\delta t_0].$$

In fact, for a sufficiently small $\varepsilon \in (0, \varepsilon_2]$ it follows from (3) that

$$x(t; \mu_0 + \varepsilon\delta\mu) \approx x_0(t) + \varepsilon\delta x(t; \delta\mu).$$

Theorem 2. *Let the following conditions hold:*
1) *The function* $f_0(z), z \in I \times O^2$ *is bounded;*
2) *There exists the limit*

$$\lim_{z \to z_0} f_0(z) = f_0^+, z \in [t_{00}, b) \times O^2$$

Then for each $\hat{t}_0 \in (t_{00}, t_{10})$ *there exist numbers* $\varepsilon_2 \in (0, \varepsilon_1)$ *and* $\delta_2 \in (0, \delta_1)$ *such that for arbitrary* $(t, \varepsilon, \delta\mu) \in [\hat{t}_0, t_{10} + \delta_2] \times (0, \varepsilon_2] \times V^+$*, where* $V^+ = \{\delta\mu \in V : \delta t_0 \geq 0\}$*, formula (3) holds, where*

$$\delta x(t; \delta\mu) = Y(t_{00}+; t)(\dot{\varphi}(t_{00}) - A(t_{00})\dot{x}(\sigma(t_{00})) - f_0^+)\delta t_0 + \beta(t; \delta\mu).$$

The following assertion is a corollary to Theorems 1 and 2.

Theorem 3. *Let the assumptions of Theorems 1 and 2 be fulfilled. Moreover,* $f_0^- = f_0^+ := \hat{f}_0$ *and* $t_{00} \notin \{\sigma(t_{10}), \sigma^2(t_{10})), ...\}$*. Then there exist numbers* $\varepsilon_2 \in (0, \varepsilon_1)$ *and* $\delta_2 \in (0, \delta_1)$ *such that for arbitrary* $(t, \varepsilon, \delta\mu) \in [t_{10} - \delta_2, t_{10} + \delta_2] \times (0, \varepsilon_2] \times V$ *formula (3) holds, where*

$$\delta x(t; \delta\mu) = Y(t_{00}; t)(A(t_{00})\dot{x}(\sigma(t_{00})) - \hat{f}_0)\delta t_0 + \beta(t; \delta\mu).$$

All assumptions of Theorem 3 are satisfied if the function $f_0(t, x, y)$ is continuous and bounded. Clearly, in this case $\hat{f}_0 = f_0(t_{00}, \varphi_0(t_{00}), \varphi_0(\tau_0(t_{00})))$.

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R E F E R E N C E S

1. Gorgodze N. On the well-posedness of the Cauchy problem for a functional differential equation taking into account variable delay perturbation. *Sem. I. Vekua Inst. Appl. Math., REPORTS*, **39** (2013), 33-37.

2. Kharatishvili G., Tadumadze T., Gorgodze N. Continuous dependence and differentiability of solution with respect to initial data and right-hand side for differential equations with deviating argument.*Mem. Diff. Eq. Math. Phys.*, **19** (2000), 3-105.

3. Ramishvili I., Tadumadze T. Formulas of variation for a solution of neutral differential equations with continuous initial condition. *Georgian Math. J.*, **11**, 1 (2004), 155-175.

4. Tadumadze T. Formulas of variation for solutions of some classes of functional differential equations and their applications. *Nonlinear Analysis*, **71** (2009) 706-710.

5 Kharatishvili G.L., Tadumadze T.A. Formulas for the variation of a solution and optimal control problems for differential equations with retarded arguments. (Russian) *Sovrem. Mat. Prilozh. No. 25, Optimal. Upr.* (2005), 3166; English transl.: *J. Math. Sci. (N. Y.)*, **140**, 1 (2007), 1175.

6. Marchuk G.I. Mathematical modelling of immune response in infection diseases. *MIA Vol. 295, Kluwer Academic Publishers, Dorbrecht,* 1994.

7. Baker C.T.H., Bocharov G.A., Paul C.A.H., Rihan F.A. Computational modelling with functional differential equations: identification, selection and sensitivity. *Applied Numerical Mathematics*, **53** (2005), 107-129.

8. Rihan F.A. Sensitivity analysis for dynamic systems with time-lags. *Journal of Computational and Applied Mathematics* **151** (2003), 445-462.

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