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EFFECTIVE SOLUTION OF THE BASIC MIXED BOUNDARY VALUE PROBLEM OF STATICS OF THE THEORY OF ELASTIC MIXTURE IN A CIRCULAR DOMAIN

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Abstract. By the method N. Muskhelishvili an explisit solution to the basic mixed boundary value problem for homogeneous equation of statics of the linear theory of elastic mixture for a circular domain is obtained.

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1. Introduction

The basic plane boundary value problem and the basic mixed boundary value problem in a simple connected domain for homogeneous equation of statics of the linear theory of elastic mixture, by analogues of general Kolosov-Muskhelishvili representation have been investigated in [3] and [2], respectively.

By the method M. Muskhelishvili an explicit solution of the basic mixed boundary value problem for homogeneous equation of statics of the linear theory of elastic mixture for an half-plane was obtained in [5].

In the present work we studied an analogous problem which in the case of the plane theory of elasticity has been studied by N. Muskhelishvili [4, §123]. To solve the problem we use the formulas due to Kolosov-Muskhelishvili and the method described in [4,5].

1. Some auxiliary formulas and operators

The homogeneous equation of static of the linear theory of elastic mixtures in a complex form is of the type [3]

$$\frac{\partial^2 U}{\partial z \partial \overline{z}} + K \frac{\partial^2 \overline{U}}{\partial \overline{z}^2} = 0, \tag{1.1}$$

where $z = x_1 + ix_2$, $\dfrac{\partial}{\partial z} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x_1} - i\dfrac{\partial}{\partial x_2}\right)$, $U = (u_1 + iu_2, u_3 + iu_4)^T$,

$u' = (u_1, u_2)^T$ and $u'' = (u_3, u_4)^T$ are partial displacements,

$$K = -\frac{1}{2}\ em^{-1}, \qquad e = \begin{bmatrix} e_4 & e_5 \\ e_5 & e_6 \end{bmatrix}, \qquad m^{-1} = \frac{1}{\triangle_0}\begin{bmatrix} m_3 & -m_2 \\ -m_2 & m_1 \end{bmatrix},$$

$$\triangle_0 = m_1 m_3 - m_2^2, \qquad m_k = e_k + \frac{1}{2}\ e_{3+k}, \quad e_1 = a_2/d_2, \quad e_2 = -c/d_2,$$

$$e_3 = a_1/d_2, \quad d_2 = a_1 a_2 - c^2, \quad a_1 = \mu_1 - \lambda_5, \quad a_2 = \mu_2 - \lambda_5, \quad c = \mu_3 + \lambda_5,$$
$$e_1 + e_4 = b/d_1, \quad e_2 + e_5 = -c_0/d_1, \quad e_3 + e_6 = a/d_1, \quad d_1 = ab - c_0^2,$$
$$a = a_1 + b_1, \quad b = a_2 + b_2, \quad c_0 = c + d, \quad b_1 = \mu_1 + \lambda_1 + \lambda_5 - \alpha_2\rho_2/\rho,$$
$$b_2 = \mu_2 + \lambda_2 + \lambda_5 + \alpha_2\rho_1/\rho, \quad \alpha_2 = \lambda_3 - \lambda_4, \quad \rho = \rho_1 + \rho_2,$$
$$d = \mu_2 + \lambda_3 - \lambda_5 - \alpha_2\rho_1/\rho \equiv \mu_3 + \lambda_4 - \lambda_5 + \alpha_2\rho_2/\rho.$$

Here $\mu_1, \mu_2, \mu_3, \quad \lambda_p, \quad p = \overline{1,5}$, are elastic modules characterizing mechanical properties of the mixture, ρ_1 and ρ_2 are partial densities of the mixture. It will be assumed that the elastic constants $\mu_1, \mu_2, \mu_3, \quad \lambda_p, \quad p = \overline{1,5}$, and partial rigid densities ρ_1 and ρ_2 satisfy the conditions (inequalities) [1].

In [3] M.O. Basheleishvili obtained the following representations:

$$U = \begin{pmatrix} u_1 + iu_2 \\ u_3 + iu_4 \end{pmatrix} = m\varphi(z) + \frac{1}{2}\; ez\overline{\varphi'(z)} + \overline{\psi(z)}, \tag{1.2}$$

$$TU = \begin{pmatrix} (Tu)_2 - i(Tu)_1 \\ (Tu)_4 - i(Tu)_3 \end{pmatrix} = \frac{\partial}{\partial S(x)}\left[(A - 2E)\varphi(z) + Bz\overline{\varphi'(z)} + 2\mu\overline{\psi(z)}\right], \tag{1.3}$$

where $\varphi(z) = (\varphi_1, \varphi_2)^T$ and $\psi(z) = (\psi_1, \psi_2)^T$are arbitrary analytic vector-functions,

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} = 2\mu m, \;\; \mu = \begin{bmatrix} \mu_1 & \mu_3 \\ \mu_3 & \mu_2 \end{bmatrix}, \; B = \mu e, \;\; m = \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix}, \;\; E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$\Delta_0 = dem > 0, \quad \Delta_1 = det\mu > 0, \quad \Delta_2 = det(A - 2E) > 0, \quad A_1 + A_3 - 2 = B_1 + B_3,$
$A_2 + A_4 - 2 = B_2 + B_4, \quad \dfrac{\partial}{\partial S(x)} = -n_2\dfrac{\partial}{\partial x_1} + n_1\dfrac{\partial}{\partial x_2}, \quad n = (n_1, n_2)^T$ unit vector of the outer normal, $(Tu)_p, \quad p = \overline{1,4}$ are stress components, $Tu = ((Tu)_1, (Tu)_2, (Tu)_3, (Tu)_4)^T$,[1,6].

Now we note that, from (1,2) we have

$$2\mu\frac{\partial U}{\partial S(x)} = \frac{\partial}{\partial S(x)}\left[A\varphi(z) + Bz\overline{\varphi'(z)} + 2\mu\overline{\psi(z)}\right]. \tag{1.4}$$

Formulas (1,2), (1,3) and (1,4) are analogous to the Kolosov-Muskhelishvili's formulas for the linear theory of elastic mixtures.

2. Statement of the mixed problem and scheme of its solution

In the present work we study an analogous problem which in the case of the plane theory of elasticity has been studied by N. Muskhelishvili [4, §123]. For the solution of the problem use will be made of the generalized Kolosov-Muskhelishvili's formula and the method developed in [4,5].

Let us assume that an elastic mixture occupies the circular domain $D^+ = \{z : |z| < 1\}$ bounded by the circumference $L = \{z : |z| = 1,\}$ and let $L_j = a_j b_j, \quad j = \overline{1,n}, \quad (a_{j+1} \neq b_j, \quad a_{n+1} \equiv a_1)$, be arcs separately lying on it, note that the points $a_1, b_1, a_2, b_2, ..., a_n, b_n$ follow each other in the positive direction on L.

Suppose that $L^{'}=\bigcup\limits_{j=1}^{n} L_j$ and $L^{''}$ is the remaining part of L.

Definition 2.1. The vector $u=(u^{'},u^{''})^T=(u_1,u_2,u_3,u_4)^T$ is called regular if (see[2]) (i) $\ u_p\in C^2(D^+)\bigcap C(D^+\bigcup L),\quad p=\overline{1,4}$;
$(ii)(Tu)_p,\ \ (p=\overline{1,4})$, is continuously extendable at every point of L from D^+ except perhaps the points a_j and b_j, $j=\overline{1,n}$;
(iii) near the points a_j and b_j, $j=\overline{1,n}$ $(Tu)_p$, $p=\overline{1,4}$ admit estimate of the type $|(Tu)_p|<const|z-\alpha_0|^{-\beta}$, $0\le\beta<1$, $z\in D^+$ $(\alpha_0=a_j;\qquad \alpha_0=b_j,\quad j=\overline{1,n})$, $p=\overline{1,4}$.

We consider the mixed boundary value problem. Define an elastic equilibrium of the plate D^+ if

$$U^+(t)=f^0(t),\qquad t\in L^{'},\qquad [TU(t)]^+=0,\quad t\in L^{''},\qquad (2,1)_{f,0}$$

where $f^0=(f_1^0,f_2^0)$ is a given complex vector-function on $L^{'}$, $(f^0(t))^{'}\in H)$. Using the Green formula [1] it is easy to prove.

Theorem 2.1. *The homogeneous mixed boundary value problem* $(2.1)_0$ *admits only a trivial solution.*

Below instead of conditions $(2.1)_{f,0}$ we consider its following equivalent conditions

$$2\mu\left(\frac{\partial U(t)}{\partial S(t)}\right)^+=\frac{\partial f(t)}{\partial S(t)},\quad t\in L^{'},\quad (TU(t))^+=0,\quad t\in L^{''},\qquad (2.1)^{'}$$

where $f(t)=2\mu f^0(t)$.

Let $t=e^{i\theta}$ $0\le\theta\le 2\pi$. Then $\dfrac{\partial}{\partial S(t)}=\dfrac{d}{d\theta}=\dfrac{d}{dt}\dfrac{d}{d\theta}=ie^{i\theta}\dfrac{d}{dt}$.

Now note that, on the basis of analogous Kolosov-Muskhelishvili's formulas (1.4) and (1.3) our problem is reduced to finding two analytic vector-functions $\phi(z)=\varphi^{'}(z)$ and $\Psi(z)=\psi^{'}(z)$ in D^+ by the boundary conditions (see $(2.1)^{'}$)

$$\begin{aligned}&[A\phi(t)+B\overline{\phi(t)}-B\overline{t\phi'(t)}-2\mu\overline{t^2\Psi(t)}]^+=f^{'}(t),\qquad t\in L^{'},\\ &[(A-2E)\phi(t)+B\overline{\phi(t)}-B\overline{t\phi'(t)}-2\mu\overline{t^2\Psi(t)}]^+=0,\qquad t\in L^{''}.\end{aligned}\qquad (2.2)$$

Consider the vector-function

$$(A-2E)\phi(z)=-B\overline{\phi(\frac{1}{\overline{z}})}+B\frac{1}{z}\overline{\phi'(\frac{1}{\overline{z}})}+2\mu\frac{1}{z^2}\overline{\Psi(\frac{1}{\overline{z}})}.\qquad (2.3)$$

From (2.3) it follows the equation (2.3) define $\phi(z)$ as an analytic vector-function toward z in the domain $|z|>1$, and to $\dfrac{1}{\overline{z}}$ in the $|z|<1$.

Due to the above formula we find that

$$2\mu\Psi(z)=(A-2E)\frac{1}{z^2}\overline{\phi(\frac{1}{\overline{z}})}+B\frac{1}{z^2}\phi(z)-B\frac{1}{z}\phi^{'}(z).\qquad (2.4)$$

If follows from (2.4) that the vector-function $\Psi(z)$ is definite in the entire $z=x_1+ix_2$ plane by means of the $\phi(z)$.

Note also that if

$$\phi_j(z) = A_0^{(j)} + A_1^{(j)} z + A_2^{(j)} z^2 + ..., \quad |z| < 1, \quad j = 1, 2,$$

$$\phi_j(z) = B_0^{(j)} + B_1^{(j)} \frac{1}{z} + B_2^{(j)} \frac{1}{z^2} + ..., \quad |z| > 1, \quad j = 1, 2,$$

then due to $A_1 + A_3 - 2 = B_1 + B_3, \quad A_2 + A_4 - 2 = B_2 + B_4, \quad (see[2])$, we can conclude that, (see(2.4)), $\Psi(z)$ to be analytic in the entire plane $z = x_1 + ix_2$ with the point $z = 0$ it is sufficient that the conditions

$$(A_0^{(1)}, A_0^{(2)})^T + (\overline{B_0^{(1)}}, \overline{B_0^{(2)}})^T = 0, \quad (B_1^{(1)}, B_1^{(2)})^T = 0 \tag{2.5}$$

be fulfilled.

In view of (2.3) the boundary conditions (2.2) can be written as:

$$\phi^+(t) - A^{-1}(A - 2E)\phi^-(t) = A^{-1} f'(t) = h(t), \quad t \in L', \quad h = (h_1, h_2)^T, \tag{2.6}$$

$$\phi^+(t) - \phi^-(t) = 0, \quad t \in L''. \tag{2.7}$$

From (2.7) it follows that the vector-function $\phi(z)$ is analytic in the entire plane $z = x_1 + ix_2$ cutting to the L'.

To solve problem (2.6) we rewrite condition (2.6) as

$$\begin{pmatrix} 1 \\ y \end{pmatrix} \phi^+(t) - \frac{2\Delta_0\Delta_1 - A_4 + A_3 y}{2\Delta_0\Delta_1} \begin{pmatrix} 1 \\ y \end{pmatrix} \phi^-(t) = \begin{pmatrix} 1 \\ y \end{pmatrix} h(t), \quad t \in L', \tag{2.8}$$

where y is an arbitrary real constant. We define the unknown y by the equation

$$y = \frac{A_2 + y(2\Delta_0\Delta_1 - A_1)}{2\Delta_0\Delta_1 - A_4 + A_3 y}, \qquad or \qquad A_3 y^2 + (A_1 - A_4)y - A_2 = 0.$$

Note that $0 < A_1 + A_4 < 4$, $A_1 + A_4 - 4\Delta_0\Delta_1 > 0$ and $(A_1 + A_4)^2 - 16\Delta_0\Delta_1 > 0)$ $(see[2])$.

On the basis of (2,8) representation we can conclude that a bounded at infinity solution of problem (2.6) is given by the formula (see [4 123])

$$\phi(z) = \frac{1}{y_2 - y_1} \begin{bmatrix} y_2 & -y_1 \\ -1 & 1 \end{bmatrix} \left[\frac{\aleph(z)}{2\pi i} \int_{L'} \frac{[\aleph^+(t)]^{-1} R(t) dt}{t - z} + \aleph(z) P_n(z) \right] \tag{2.10}$$

where y_1 and y_2 are the roots of equation (2.9),

$$R(t) = \begin{bmatrix} 1 & y_1 \\ 1 & y_2 \end{bmatrix} h(t), \qquad \aleph(z) = \begin{bmatrix} \aleph_1(z) & 0 \\ 0 & \aleph_2(z) \end{bmatrix},$$

$$\aleph_j(z) = \prod_{k=1}^{n} (z - a_k)^{-\frac{1}{2} - i\beta_j} (z - b_k)^{-\frac{1}{2} + i\beta_j}, \qquad \beta_j = \frac{ln|M_j|}{2\pi},$$

$$M_j = \frac{1}{4\Delta_0\Delta_1}\left[4\Delta_0\Delta_1 - A_1 - A_4 - (-1)^j\sqrt{(A_1+A_4)^2 - 16\Delta_0\Delta_1}\right] < 0$$

$$P_n(z) = (P_{n_1}(z), P_{n_2}(z))^T, \qquad P_{n_j}(z) = \sum_{q=0}^{n} C_q^{(j)} z^{n-q}, \quad j = 1,2.$$

To define $C_q^{(j)}$, $\quad j = 1,2$, $\quad q = \overline{0,n}$, we use the following conditions (see [4, 123], $(2.1)'$ and (2.5))

$$2\mu \int\limits_{b_k a_{k+1}} d\begin{bmatrix} u_1 + iu_2 \\ u_3 + iu_4 \end{bmatrix} = f(a_{k+1}) - f(b_k), \qquad \phi(0) + \overline{\phi(\infty)} = 0. \tag{2.11}$$

If we take into account (2.6), (2.7) and (2.10) for determining the unknown vectors $(C_q^1, C_q^2)^T$, $\quad q = \overline{0,n}$, from (2.11) we obtain the following system of equations:

$$2\int\limits_{b_k a_{k+1}} \phi_0(t_0)dt_0 + \sum_{q=0}^{n} N_{kq}\begin{pmatrix} C_q^{(1)} \\ C_q^{(2)} \end{pmatrix} = f(a_{k+1}) - f(b_k), \tag{2.12}$$

$$\begin{pmatrix} \overline{C}_0^{(1)} \\ \overline{C}_0^{(2)} \end{pmatrix} + \aleph(0)\begin{pmatrix} Cn^{(1)} \\ Cn^{(2)} \end{pmatrix} + \frac{\aleph(0)}{2\pi i}\int\limits_{L'} [\aleph^+(t)]^{-1}\frac{R(t)dt}{t} = 0. \tag{2.13}$$

where (see (2.10))

$$\phi_0(t) = \frac{1}{y_0 - y_1}\begin{bmatrix} y_2 & -y_1 \\ -1 & 1 \end{bmatrix}\frac{\aleph(t_0)}{2\pi i}\int\limits_{L'} [\aleph^+(t)]^{-1}\frac{R(t)dt}{t - t_0},$$

$$N_{kq} = \frac{2}{y_2 - y_1}\begin{bmatrix} y_2 & -y_1 \\ -1 & 1 \end{bmatrix}\int\limits_{b_k a_{k+1}} \aleph(t)t^{n-q}dt.$$

Now note that, on the basis of the uniqueness theorem (see Theorem 2.1) for (2.1) mixed problem, we can conclude that the (2.12) and (2.13) system is solvable for $C_q^{(1)}$, $\quad q = \overline{0,n}$, $\quad j = 1,2$.

Having found $C_q^{(1)}$, $\quad q = \overline{0,n}$, $\quad j = 1,2$ we can be define $\phi(z)$, hence $\Psi(z)$, $\varphi(z)$ and $\psi(z)$. Finally by (1.2) we obtain the solution of the mixed $(2.1)_{f,0}$ problem.

The mixed boundary value problem considered in the paper, for domain outside the circle, can be solved in a similar way.

R E F E R E N C E S

1. Basheleishvili M. Two-dimensional boundary value problems of hte theory of elastic mixtures. *Mem. Differential Equations Math. Phys.*, **6**(1995), 59-105.

2. Bashaleisivili M., Zazashvili Sh. The basic mixed plane boundary value problem of statics in the elastic mixture theory. *Georgian Math J.*, **7**, 3 (2000), 427-440.

3. Bashaleishvili M., Svanadze K. A new method of solving the basic plane bounday value problems of statics of the elastic mixture theory. *Georgian Math. J*, **8**, 3 (2001), 427-446.

4. Muskhelishvili N. Some basic problem of the mathematical theory of Elasticity. (Russian) *Nauka, Moscow*, 1966.

5. Svanadze K. Solution of the basic mixed boundary value problem of statics in the linear theory of elastic mixture for an half-plane. *Reports of Enlarged Session of the Seminar of I. Vekua Institute of Applied Mathematics*, **24** (2010), 122-125.

6. Svanadze K. Solution of a mixed boundary value problem of the plane theory of elastic mixture for a multiply connected domain with a partially unknown boundary having the axis of symmetry. *Pros. of A. Razmadze Math. Inst.*, **55** (2011), 73-90.

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