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ON OSCILLATORY PROPERTIES OF SOLUTIONS OF n-TH ORDER GENERALIZED EMDEN-FOWLER DIFFERENTIAL EQUATIONS WITH DELAY ARGUMENT

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Abstract. In the paper the following differential equation

$$u^{(n)}(t) + p(t)\left|u(\tau(t))\right|^{\mu(t)} \operatorname{sign} u(\tau(t)) = 0$$

is considered, where $n \geq 3$, $p \in L_{\mathrm{loc}}(R_+; R_-)$, $\mu \in C(R_+;(0,+\infty))$, $\tau \in C(R_+;R_+)$, $\tau(t) \leq t$ for $t \in R_+$ and $\lim\limits_{t\to+\infty} \tau(t) = +\infty$. We say that the equation is "almost linear" if the condition $\lim\limits_{t\to+\infty} \mu(t) = 1$ is fulfilled, while if $\limsup\limits_{t\to+\infty} \mu(t) \neq 1$ or $\liminf\limits_{t\to+\infty} \mu(t) \neq 1$, then the equation is an essentially nonlinear differential equation. In case of "almost linear" and essentially nonlinear differential equations to have Property **A** have been extensively studied [1–5]. In the paper new sufficient conditions are established for a general class of essentially nonlinear functional differential equations to have Property **B**.

Keywords and phrases: Property **B**, oscillation, functional differential equation.

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1. Introduction

This work deals with the investigation of oscillatory properties of solutions of a functional-differential equation of the form

$$u^{(n)}(t) + p(t)\left|u(\tau(t))\right|^{\mu(t)} \operatorname{sign} u(\tau(t)) = 0, \tag{1.1}$$

where

$$\begin{gathered} p \in L_{\mathrm{loc}}(R_+; R_-), \quad \mu \in C(R_+;(0,+\infty)), \\ \tau \in C(R_+;R_+), \quad \tau(t) \leq t \quad \text{and} \quad \lim_{t\to+\infty} \tau(t) = +\infty. \end{gathered} \tag{1.2}$$

It will always be assumed that the condition

$$p(t) \leq 0 \quad \text{for} \quad t \in R_+ \tag{1.3}$$

is fulfilled.

Let $t_0 \in R_+$. A function $u : [t_0,+\infty)$ is said to be a proper solution of equation (1.1) if it is locally absolutely continuous together with its derivatives up to order $n-1$ inclusive, $\sup\{|u(s)| : s \geq t\} > 0$ for $t \geq t_0$ and there exists a function $\overline{u} \in C(R_+;R)$ such that $\overline{u}(t) \equiv u(t)$ on $[t_0,+\infty)$ and the equality $\overline{u}^{\,(n)}(t) + p(t)|\overline{u}(\tau(t))|^{\mu(t)} \operatorname{sign} \overline{u}(\tau(t)) = 0$ holds almost everywhere for $t \in [t_0,+\infty)$. A proper solution $u : [t_0,+\infty) \to R$ of equation (1.1) is said to be oscillatory if it has a sequence of zeros tending to $+\infty$. Otherwise the solution u is said to be nonoscillatory.

Definition 1.1. We say that equation (1.1) has Property **A** if any of its proper solutions is oscillatory when n is even, and either is oscillatory or satisfies

$$\left|u^{(i)}(t)\right| \downarrow 0 \quad \text{as} \quad t \uparrow +\infty \quad (i = 0, \dots, n-1) \tag{1.4}$$

when n is odd.

Definition 1.2. We say that equation (1.1) has Property **B** if any of its proper solutions is either oscillatory or satisfies either (1.4) or

$$\left|u^{(i)}(t)\right| \uparrow +\infty \quad \text{as} \quad t \uparrow +\infty \quad (i = 0, \dots, n-1) \tag{1.5}$$

when n is even and either is oscillatory or satisfies (1.5), when n is odd.

Definition 1.3. We say that equation (1.1) is almost linear if the condition $\lim\limits_{t\to+\infty} \mu(t) = 1$ holds, while if $\liminf\limits_{t\to+\infty} \mu(t) \neq 1$ or $\limsup\limits_{t\to+\infty} \mu(t) \neq 1$, then we say that the equation is an essentially nonlinear differential equation.

Oscillatory properties of almost linear and essentially nonlinear differential equation with advanced argument are studied well enough in [1–5]. For Emden-Fowler equations with deviating arguments, essential contribution was made in [6–9]. In the present paper for the generalized differential equation with delay argument, sufficient conditions are established for equation (1.1) to have Property **B**. Analogously results for Property **A**, see [10].

2. Essentially nonlinear differential equation with property B

The following notations will be used throughout the work

$$\begin{gathered}\alpha = \inf\left\{\mu(t) : t \in R_+\right\}, \quad \beta = \sup\left\{\mu(t) : t \in R_+\right\}, \\ \tau_{(-1)}(t) = \sup\left\{s \geq 0,\ \tau(s) \leq t\right\}, \quad \tau_{(-k)} = \tau_{(-1)} \circ \tau_{(-(k-1))}, \quad k = 2, 3, \dots\end{gathered} \tag{2.1}$$

Clearly $\tau_{(-1)}(t) \geq t$ and $\tau_{(-1)}$ is nondecreasing and coincides with the inverse of τ when the latter exists.

Let $\alpha \in [1, +\infty)$, $\gamma \in (1, +\infty)$, $\ell \in \{1, \dots, n-2\}$ and $t_* \in R_+$. Denote

$$\rho^{(\alpha)}_{1,\ell,t_*}(t) = \ell! \, \exp\left\{\gamma_\ell(\alpha) \int_{\tau_{(-1)}(t_*)}^{t} \int_{s}^{+\infty} \xi^{n-\ell-2} (\tau(\xi))^{1+(\ell-1)\mu(\xi)} |p(\xi)| d\xi \, ds\right\}, \tag{2.2}$$

$$\begin{aligned}\rho^{(\alpha)}_{i,\ell,t_*}(t) = \ell! + \frac{1}{(n-\ell)!} \int_{\tau_{(-i)}(t_*)}^{t} \int_{s}^{+\infty} \xi^{n-\ell-1} (\tau(\xi))^{(\ell-1)\mu(\xi)} \times \\ \times \left(\frac{1}{\ell!} \rho^{\alpha)}_{i-1,\ell,t_*}(\tau(\xi))\right)^{\mu(\xi)} |p(\xi)| d\xi \, ds \quad (i = 2, 3, \dots),\end{aligned} \tag{2.3}$$

$$\gamma_\ell(\alpha) = \begin{cases} \gamma & \text{if} \quad \alpha > 1, \\ \dfrac{1}{\ell! \, (n-\ell)!} & \text{if} \quad \alpha = 1. \end{cases} \tag{2.4}$$

In the section, when $\alpha > 1$, we derive sufficient conditions for functional differential equation (1.1) to have Property **B**.

Proposition 2.1. *Let $\alpha > 1$, conditions* (1.2) *and* (1.3) *be fulfilled and for any $\ell \in \{1,\dots,n\}$ with $\ell + n$ even, the conditions*

$$\int_0^{+\infty} t^{n-\ell}\big(c,\tau^{\ell-1}(t)\big)^{\mu(t)}|p(t)|dt = +\infty \ \ \textit{for} \ \ c \in (0,1] \tag{$2.5_{\ell,c}$}$$

and

$$\int_0^{+\infty} t^{n-\ell-1}\big(\tau(t)\big)^{\ell\,\mu(t)}|p(t)|dt = +\infty \ \ \textit{for} \ \ \ell \in \{1,\dots,n-2\} \tag{2.6_ℓ}$$

be fulfilled. Moreover, let for any large $t_* \in R$, for some $k \in N$, $\gamma \in (1,+\infty)$ and $\delta \in (1,\alpha]$

$$\int_{\tau_{(-k)}(t_*)}^{+\infty}\int_s^{+\infty} \xi^{n-\ell-1-\delta}(\tau(\xi))^{\delta+(\ell-1)\mu(\xi)}\Big(\frac{1}{\ell!}\,\rho^{(\alpha)}_{k,\ell,t_*}(\tau(\xi))\Big)^{\mu(\xi)-\delta}|p(\xi)|d\xi\,ds = +\infty. \tag{2.7_ℓ}$$

Then equation (1.1) *has Property* **B***, where α is defined by first condition of* (2.1) *and $\rho^{(\alpha)}_{k,\ell,t_*}$ is given by* (2.2)–(2.4).

Proposition 2.1′. *Let $\alpha > 1$, $\beta < +\infty$, conditions* (1.2) *and* (1.3) *be fulfilled and for any $\ell \in \{1,\dots,n-2\}$ with $\ell + n$ even, conditions* ($2.5_{\ell,1}$) *and* (2.6_ℓ) *hold. Moreover, let for some $k \in N$, $\gamma \in (1,+\infty)$ and $\delta \in (1,\alpha]$ condition* (2.7_ℓ) *be fulfilled. Then equation* (1.1) *has Property* **B***, where α and β are defined by* (2.1) *and $\rho^{(\alpha)}_{k,\ell,t_*}$ is given by* (2.2)–(2.4).

Theorem 2.1. *Let $\alpha > 1$, conditions* (1.2), (1.3), ($2.5_{1,c}$) *and*

$$\liminf_{t\to+\infty} \frac{(\tau(t))^{\mu(t)}}{t} > 0 \tag{2.8}$$

be fulfilled. Moreover, let for some $\delta \in (1,\alpha]$ the conditions

$$\int_0^{+\infty}\int_s^{+\infty} \xi^{n-2-\delta}(\tau(\xi))^{\delta}|p(\xi)|d\xi\,ds = +\infty, \tag{2.9}$$

when n is odd and

$$\int_0^{+\infty}\int_s^{+\infty} \xi^{n-3-\delta}(\tau(\xi))^{\delta+\mu(\xi)}|p(\xi)|d\xi\,ds = +\infty, \tag{2.10}$$

when n is even, be fulfilled. Then equation (1.1) *has Property* **B***, where α is defined by the first condition of* (2.1).

Theorem 2.1′. *Let $\alpha > 1$ and $\beta < +\infty$, conditions* (1.2), (1.3), ($2.5_{1.1}$), (2.6_1) *and* (2.8) *be fulfilled. Moreover, let for some $\delta \in (1,\alpha)$, when n is odd (n is even) condition* (2.9) ((2.10)) *holds. Then equation* (1.1) *has Property* **B***, where α and β are given by* (2.1).

Remark 2.1. In Theorem 2.1 condition ($2.5_{1,c}$) cannot be replaced by condition ($2.5_{1,1}$). Indeed, let $n \geq 3$, $c \in (0,1)$, $c_1 \in (c,1)$,

$$\mu(t) = n\log_{\frac{1}{c_1}} t, \ \ p(t) = -\frac{c\,n!}{t^{1+n}}\,c^{-\mu(t)}\bigg(t^{n-1} + \frac{(-1)^n}{t}\bigg)^{-\mu(t)} \ \ \text{and} \ \ \tau(t) \equiv t.$$

It is obvious that condition $(2.5_{1,1})$ is fulfilled, but for large t, equation (1.1) has the solution $u(t) = c(t^{n-1} + \frac{(-1)^n}{t})$. Therefore, equation (1.1) has the solution u, satisfying the condition $\lim\limits_{t\to+\infty} u^{(n-1)}(t) = c(n-1)!$, that is equation (1.1) does not have Property **B**.

Theorem 2.2. *Let* $\alpha > 1$, *let conditions* (1.2), (1.3), $(2.5_{1,c})$, (2.6_1) *and* (2.8) *be fulfilled and*

$$\liminf_{t\to+\infty} t \int_t^{+\infty} s^{n-3}\tau(s)|p(s)|ds > 0. \tag{2.11}$$

Moreover, let for some $\delta \in (1, \alpha]$ *and* $\gamma > 0$

$$\int_0^{+\infty} \int_s^{+\infty} \xi^{n-2-\delta}(\tau(\xi))^{\delta+\gamma(\mu(\xi)-\delta)}|p(\xi)|d\xi\, ds = +\infty. \tag{2.12}$$

Then equation (1.1) *has Property* **B**, *where* α *is defined by the first condition of* (2.1).

Theorem 2.2′. *Let* $\alpha > 1$ *and* $\beta < +\infty$, *conditions* (1.2), (1.3), $(2.5_{1.1})$, (2.6_1), (2.8) *and* (2.11) *be fulfilled. Moreover, if for some* $\delta \in (1, \alpha]$ *and* $\gamma > 0$, *condition* (2.12) *holds, then equation* (1.1) *has Property* **B**, *where* α *and* β *are given by* (2.1).

Theorem 2.3. *Let* $\alpha > 1$, *conditions* (1.2), (1.3), $(2.5_{1,c})$, (2.6_1), (2.8) *and* (2.11) *be fulfilled. Moreover, if there exists* $m \in N$ *such that*

$$\liminf_{t\to+\infty} \frac{\tau^m(t)}{t} > 0, \tag{2.13}$$

then equation (1.1) *has Property* **B**, *where* α *is given by the first condition of* (2.1).

Theorem 2.3′. *Let* $\alpha > 1$ *and* $\beta < +\infty$, *conditions* (1.2), (1.3), $(2.5_{1.1})$, (2.6_1), (2.8), (2.11) *and for some* $m \in N$ *condition* (2.13) *be fulfilled. Then equation* (1.1) *has Property* **B**, *where* α *and* β *are given by* (2.1).

Theorem 2.4. *Let* $\alpha > 1$, *conditions* (1.2), (1.3), $(2.5_{n-1,c})$, (2.6_{n-1}) *and*

$$\limsup_{t\to+\infty} \frac{(\tau(t))^{\mu(t)}}{t} < +\infty \tag{2.14}$$

be fulfilled. Moreover, if for some $\delta \in (1, \alpha]$

$$\int_0^{+\infty} \int_s^{+\infty} \xi^{1-\delta}(\tau(\xi))^{\delta+(n-3)\mu(\xi)}|p(\xi)|d\xi\, ds = +\infty, \tag{2.15}$$

then equation (1.1) *has Property* **B**, *where* α *is given by the first condition of* (2.1).

Theorem 2.4′. *Let* $\alpha > 1$ *and* $\beta < +\infty$, *conditions* (1.2), (1.3), $(2.5_{n-1,1})$, (2.6_{n-1}) *and* (2.14) *be fulfilled. Moreover, if for some* $\delta \in (1, \alpha]$ *condition* (2.15) *holds, then equation* (1.1) *has Property* **B**, *where* α *and* β *are given by* (2.1).

Theorem 2.5. *Let* $\alpha > 1$, *conditions* (1.2), (1.3), $(2.5_{n-1,c})$, (2.7_{n-1}) *and* (2.14) *be fulfilled and*

$$\liminf_{t\to+\infty} t \int_t^{+\infty} (\tau(s))^{1+(n-3)\mu(s)}|p(s)|ds > 0. \tag{2.16}$$

Moreover, if for some $\delta \in (1, \alpha]$ *and* $\gamma > 0$

$$\int_0^{+\infty} \int_s^{+\infty} \xi^{1-\delta} (\tau(\xi))^{\delta+(n-3)\mu(\xi)+\gamma(\mu(\xi)-\delta)} |p(\xi)| d\xi \, ds = +\infty, \tag{2.17}$$

then equation (1.1) *has Property* **B**, *where* α *is given by* (2.1).

Theorem 2.5′. *Let* $\alpha > 1$ *and* $\beta < +\infty$, *conditions* (1.2), (1.3), $(2.5_{n-1,1})$, (2.6_{n-1}), (2.14) *and* (2.16) *be fulfilled and for some* $\delta \in (1, \alpha)$ *and* $\gamma > 0$ *condition* (2.17) *holds. Then equation* (1.1) *has Property* **B**, *where* α *and* β *are given by* (2.1).

Theorem 2.6 *Let* $\alpha > 1$, *conditions* (1.2), (1.3), $(2.5_{n-1,c})$, (2.6_{n-1}), (2.14) *and* (2.17) *be fulfilled. Moreover, if for some* $m \in N$ *condition* (2.13) *holds, then equation* (1.1) *has Property* **B**, *where* α *is given by* (2.1).

Theorem 2.6′. *Let* $\alpha > 1$ *and* $\beta < +\infty$, *conditions* (1.2), (1.3), $(2.5_{n-1,1})$, (2.6_{n-1}) *and* (2.17) *be fulfilled. Moreover, if for some* $m \in N$ *condition* (2.13) *holds, then equation* (1.1) *has Property* **B**, *where* α *and* β *are given by* (2.1).

3. Quasi-linear differential equations with property B

In the section we define sufficient conditions for functional differential equations (1.1), when $\alpha = 1$, to have Property **B**.

Proposition 3.1 *Let* $\alpha = 1$, *conditions* (1.2) *and* (1.3) *be fulfilled and for any* $\ell \in \{1, \dots, n+1\}$ *with* $\ell + n$ *even, conditions* $(2.5_{\ell,c})$ *and* (2.6_ℓ) *hold. Let moreover, for any large* $t_* \in R_+$ *and for some* $k \in N$

$$\limsup_{t \to +\infty} \frac{1}{t} \int_{\tau_{(-k)}(t_*)}^{t} \int_s^{+\infty} \xi^{n-\ell-1} \big(\tau(\xi)\big)^{(\ell-1)\mu(t)} \times \\ \times \left(\frac{1}{\ell!} \rho^{(1)}_{k,\ell,t_*}(\tau(\xi)) \right)^{\mu(\xi)} |p(\xi)| d\xi \, ds > 0. \tag{3.1_ℓ}$$

Then equation (1.1) *has Property* **B**, *where* α *is given by the first condition of* (2.1).

Proposition 3.1′. *Let* $\alpha = 1$ *and* $\beta < +\infty$, *conditions* (1.2) *and* (1.3) *be fulfilled and for any* $\ell \in \{1, \dots, n\}$ *with* $\ell + n$ *even, conditions* $(2.5_{\ell,1})$ *and* (2.6_ℓ) *hold. Moreover, let for any large* $t_* \in R_+$ *and for some* $k \in N$, *condition* (3.1_ℓ) *holds. Then equation* (1.1) *has Property* **B**, *where* α *and* β *are given by* (2.1).

Theorem 3.1 *Let* $\alpha = 1$, *conditions* (1.2), (1.3), $(2.5_{1,c})$, (2.6_1) *and* (2.8) *be fulfilled and*

$$\limsup_{t \to +\infty} \frac{1}{t} \int_0^t \int_s^{+\infty} \xi^{n-2} |p(\xi)| d\xi \, ds > 0. \tag{3.2}$$

Then equation (1.1) *has Property* **B**, *where* α *is defined by first condition of* (2.1).

Theorem 3.1′. *Let* $\alpha = 1$ *and* $\beta < +\infty$, *conditions* (1.2), (1.3), $(2.5_{1,1})$, (2.6_1), (2.8) *and* (3.2) *be fulfilled. Then equation* (1.1) *has Property* **B**, *where* α *and* β *are given by* (2.1).

Theorem 3.2 *Let* $\alpha = 1$, *conditions* (1.2), (1.3), $(2.5_{1,c})$, (2.6_1) *be fulfilled. Let moreover*

$$\liminf_{t \to +\infty} \frac{(\tau(t))^{\mu(t)}}{t} > 1 \tag{3.3}$$

and

$$\liminf_{t\to+\infty} t\int_t^{+\infty} s^{n-3}\tau(s))ds > (n-1)!. \qquad (3.4)$$

Then for equation (1.1) *to have Property* **B** *it is sufficient that*

$$\limsup_{t\to+\infty} \frac{1}{t}\int_0^t\int_s^{+\infty} \xi^{n-2}(\tau(\xi))^{\mu(\xi)}|p(\xi)|d\xi\, ds > 0. \qquad (3.5)$$

Theorem 3.2$'$. *Let* $\alpha = 1$ *and* $\beta < +\infty$*, conditions* (1.2)*,* (1.3)*,* $(2.5_{1,1})$*,* (2.6_1)*,* (3.3) *and* (3.4) *be fulfilled. Then equation* (1.1) *has Property* **B***, it is sufficient that condition* (3.5) *holds.*

Theorem 3.3 *Let* $\alpha = 1$*, conditions* (1.2)*,* (1.3)*,* $(2.5_{n-1,c})$*,* (2.6_{n-2}) *be fulfilled. Moreover, if the conditions*

$$\liminf_{t\to+\infty} \frac{(\tau(t))^{\mu(t)}}{t} < 1 \qquad (3.6)$$

and

$$\liminf_{t\to+\infty} t\int_t^{+\infty} (\tau(s))^{1+(n-3)\mu(s)}|p(s)|ds > 2(n-2)! \qquad (3.7)$$

are fulfilled, then for equation (1.1) *to have Property* **B** *it is sufficient that*

$$\limsup_{t\to+\infty} \frac{1}{t}\int_0^t\int_s^{+\infty} \xi(\tau(\xi))^{(n-3)\mu(\xi)}(\tau(\xi))^{\mu(\xi)}|p(\xi)|d\xi\, ds > 0. \qquad (3.8)$$

Theorem 3.3$'$. *Let* $\alpha = 1$ *and* $\beta < +\infty$*, conditions* (1.2)*,* (1.3)*,* $(2.5_{n-1,1})$*,* (2.6_{n-1})*,* (3.6) *and* (3.7) *be fulfilled. Then for equation* (1.1) *to have Property* **B***, it is sufficient that condition* (3.8) *holds.*

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