Seminar of I. Vekua Institute of Applied Mathematics
REPORTS, Vol. 40, 2014

OSCILLATION CRITERIA FOR DIFFERENCE EQUATIONS WITH SEVERAL DELAY ARGUMENTS

Koplatadze R.

Abstract. In the paper the following difference equation

$$\Delta u(k) + \sum_{i=1}^{m} p_i(k)\, u(\tau_i(k)) = 0$$

is considered, where $m \in N$, the functions $p_i : N \to R_+$, $\tau_i : N \to N$, $\tau_i(k) \le k-1$, $\lim\limits_{k\to+\infty} \tau_i(k) = +\infty$ $(i = 1,\dots,m)$ are defined on the set of natural numbers and the difference operator is defined by $\Delta u(k) = u(k+1) - u(k)$. New oscillation criteria of all solutions to these equation are established.

Keywords and phrases: Oscillation, proper solution, difference equations with several delay.

AMS subject classification (2010): 34K11.

1. Introduction

Consider the difference equation

$$\Delta u(k) + \sum_{i=1}^{m} p_i(k)\, u(\tau_i(k)) = 0, \tag{1.1}$$

where $m \ge 1$ is a natural number, $p_i : N \to R_+$, $\tau_i : N \to N$, $(i = 1,\dots,m)$, are functions defined on the set $N = \{1, 2, \dots\}$ and $\Delta u(k) = u(k+1) - u(k)$. Everywhere below it is assumed that

$$\lim_{k\to+\infty} \tau_i(k) = +\infty, \quad \tau_i(k) \le k-1. \tag{1.2}$$

For each $n \in N$ denote $N_n = \{n, n+1, \dots\}$.

Definition 1.1. Let $n \in N$. We will call a function $u : N \to R$ a proper solution of equation (1.1) on the set N_n, if it satisfies (1.1) on N_n and $\sup\{|u(i)| : i \ge k\} > 0$ for any $k \in N_n$.

Definition 1.2. We say that a proper solution $u : N_n \to R$ of equation (1.1) is oscillatory if for any $k \in N$ there exist $n_1, n_2 \in N_k$ such that $u(n_1) \cdot u(n_2) \le 0$. Otherwise the solution is called nonoscillatory.

Definition 1.3. Equation (1.1) is said to be oscillatory, if any of its proper solutions is oscillatory.

The problem of oscillation of solutions of linear difference equation (1.1) for $m = 1$, has been studied by several authors, see [1,2] and references therein.

As to investigation of the analogous problem for equation of type (1.1) $(m > 1)$, to our knowledge for them there have not been obtained results analogous to those known for equation (1.1), where $m = 1$. Analogous results for first order differential equations with several delay see [3,4].

2. Sufficient conditions for oscillation

Denote

$$\psi_1(k) = 1, \quad \psi_s(k) = \left(\prod_{\ell=1}^{m}\prod_{j=\tau_\ell(k)}^{k}\left[1 + m\Big(\prod_{\ell=1}^{m} p_\ell(j)\Big)^{\frac{1}{m}}\psi_{s-1}(j)\right]\right)^{\frac{1}{m}} \tag{2.1}$$

$$k \in N, \quad s = 2, 3, \dots$$

Theorem 2.1. *Let there exist $k_0 \in N$ and nondecreasing functions $\sigma_i : N \to N$ $(i = 1, \dots, m)$ such that*

$$1 + \tau_i(k) \le \sigma_i(k) \le k \quad \textit{for} \quad k \in N \quad (i = 1, \dots, m) \tag{2.2}$$

and

$$\limsup_{k\to+\infty}\prod_{\ell=1}^{m}\left(\prod_{i=1}^{m}\sum_{s=\sigma_\ell(k)}^{k} p_i(s)\prod_{j=\tau_i(s)}^{\sigma_i(k)-1}\left[1 + m\Big(\prod_{\ell=1}^{m} p_\ell(j)\Big)^{\frac{1}{m}}\psi_{k_0}(j)\right]\right)^{\frac{1}{m}} > \frac{1}{m^m}\,,$$

then equation (1.1) *is oscillatory, where ψ_{k_0} is given by* (2.1) *when $k = k_0$.*

Corollary 2.1. Let there exist nondecreasing functions $\sigma_i : N \to R$ such that

$$\limsup_{k\to+\infty}\prod_{\ell=1}^{m}\left(\prod_{i=1}^{m}\sum_{s=\sigma_\ell(k)}^{k} p_i(s)\prod_{j=\tau_i(s)}^{\sigma_i(k)-1}\left[1 + m\Big(\prod_{\ell=1}^{m} p_\ell(j)\Big)^{\frac{1}{m}}\right]\right)^{\frac{1}{m}} > \frac{1}{m^m}\,,$$

then equation (1.1) is oscillatory.

Corollary 2.2. Let there exist nondecreasing functions $\sigma_i : N \to R$ such that condition (2.2) is fulfilled and

$$\limsup_{k\to+\infty}\prod_{\ell=1}^{m}\left(\prod_{i=1}^{m}\sum_{s=\sigma_\ell(k)}^{k} p_i(s)\right)^{\frac{1}{m}} > \frac{1}{m^m}\,,$$

then equation (1.1) is oscillatory.

Theorem 2.2. *Let there exist nondecreasing functions $\sigma_i : N \to N$ such that* (2.2) *is fulfilled,*

$$\limsup_{k\to+\infty}\prod_{\ell=1}^{m}\left(\prod_{i=1}^{m}\sum_{s=\sigma_\ell(k)}^{k} p_i(s)\prod_{j=\tau_j(s)}^{\sigma_i(k)-1}\Big(\prod_{\ell=1}^{m} p_\ell(j)\Big)^{\frac{1}{m}}\right)^{\frac{1}{m}} > 0 \tag{2.3}$$

and

$$\liminf_{k\to+\infty}\prod_{j=\tau_\ell(k)}^{k}\left(\prod_{i=1}^{m} p_i(j)\right)^{\frac{1}{m}} = \alpha_\ell > 0 \quad (\ell = 1, \dots, m). \tag{2.4}$$

Moreover, if for some $\ell \in \{1,\dots,m\}$

$$\lim_{k\to+\infty}\big(k-\tau_\ell(k)\big)=+\infty, \tag{2.5}$$

then equation (1.1) *is oscillatory.*

Theorem 2.3. *Let there exist nondecreasing functions* $\sigma_i : N \to N$*, such that* (2.2) *and* (2.3) *hold. If moreover,*

$$\liminf_{k\to+\infty}\big(k-\tau_\ell(k)\big)=n_\ell\in N \quad (\ell=1,\dots,m) \tag{2.6}$$

and

$$\prod_{\ell=1}^{m}\alpha_\ell>\frac{1}{n_0^m}\left(\frac{n_0}{n_0+m}\right)^{n_0+m}, \tag{2.7}$$

where α_ℓ $(\ell=1,\dots,m)$ *are given by* (2.4) *and* $n_0=\sum\limits_{\ell=1}^{m}n_\ell$*. Then equation* (1.1) *is oscillatory.*

Theorem 2.4. *Let* $\tau_i : N\to N$ $(i=1,\dots,m)$ *be nondecreasing functions, let* (2.6) *and* (2.7) *be fulfilled and*

$$\liminf_{k\to+\infty}\sum_{i=\tau_j(k)}^{k-1}p_j(i)>0 \quad (j=1,\dots,m). \tag{2.8}$$

Then equation (1.1) *is oscillatory, where* α_ℓ *is given by* (2.4).

Theorem 2.5. *Let there exist nondecreasing functions* $\sigma_i : N\to N$ *such that* (2.2), (2.3) *and let* (2.6) *be fulfilled. Moreover, if* $m\le\sum\limits_{\ell=1}^{m}n_\ell$ *and*

$$\prod_{\ell=1}^{m}\alpha_\ell>\big(2\sqrt{m}\big)^{-\sum\limits_{\ell=1}^{m}(n_\ell+1)}, \tag{2.9}$$

then equation (1.1) *is oscillatory, where*

$$\alpha_\ell=\liminf_{k\to+\infty}\prod_{j=\tau_\ell(k)}^{k}\left(\prod_{i=1}^{m}p_i(j)\right)^{\frac{1}{2m}} \quad (\ell=1,\dots,m). \tag{2.10}$$

Theorem 2.6. *Let* $\tau_i : N\to N$ *be nondecreasing functions and* (2.6), (2.8) *and let* (2.9) *be fulfilled. Then equation* (1.1) *is oscillatory, where* α_ℓ *is given by* (2.10).

Acknowledgement. The work was supported by the Sh. Rustaveli National Science Foundation. Grant No 31/09.

R E F E R E N C E S

1. Gyori I., Ladas G. Oscillation theory of delay differential equations with applications. *Clarendon press, Oxford*, 1991.

2. Koplatadze R., Pinelas S. Oscillation criteria for first order linear difference equations with several delay arguments. *Nonlinear Oscillations*, **17**, 2 (2014), 247-267.

3. Infante G., Koplatadze R., Stavroulakis I. Oscillation criteria for differential equations with several nonmonotone arguments. *Funkcialaj Ekvacioj*, (accepted).

4. Koplatadze R. The specific properties of solutions of first order differential equations with several delay argument. *Journal of Contemporary Mathematical Analysis*, (accepted).

Received 20.06.2014; revised 17.07.2014; accepted 19.08.2014.

Author's address:

R. Koplatadze
I. Vekua Institute of Applied Mathematics of
Iv. Javakhishvili Tbilisi State University
2, University St., Tbilisi 0186
Georgia
E-mail: roman.koplatadze@tsu.ge