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EFFECTIVE SOLUTION OF THE DIRICHLET BVP OF THERMOELASTICITY WITH MICROTEMPERATURES FOR AN ELASTIC SPACE WITH A SPHERICAL CAVITY

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Abstract. In the present paper the linear theory of thermoelasticity with microtemperatures is considered. The representation of regular solution for the equations of steady vibration of the 3D theory of thermoelasticity with microtemperatures is obtained. We use it for explicitly solving Dirichlet boundary value problem (BVP) for an elastic space with a spherical cavity. The obtained solutions are represented as absolutely and uniformly convergent series.

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1. Introduction

A thermodynamic theory for materials with inner structure whose particles, in addition to the classical displacement and temperature fields, possess microtemperatures was established by Grot [1]. The linear theory of thermoelasticity with microtemperatures was presented in [2], where the existence theorems were proved and the continuous dependence of solutions of the initial data and body loads were established. The fundamental solutions of the equations of the three-dimensional (3D) theory of thermoelasticity with microtemperatures were constructed by Svanadze [3]. The representations of the Galerkin type and general solutions of the system in this theory were obtained by Scalia, Svanadze and Tracinà [4]. The 3D linear theory of thermoelasticity for microstretch elastic materials with microtemperatures was constructed by Iesan [5], where the uniqueness and existence theorems in the dynamical case for isotropic materials are proved. A wide class of external BVPs of steady vibrations is investigated by Svanadze [6]. Effective solution of the Dirichlet and the Neumann BVPs of the linear theory of thermoelasticity with microtemperatures for a spherical ring are obtained in [7-8].

The two-dimensional model of thermoelasticity with microtemperatures is considered by Basheleishvili, Bitsadze and Jaiani in [9,10,11,12]. In particular, fundamental and singular solutions of the system of equations of the equilibrium of the 2D thermoelastisity theory with microtemperatures were constructed. Uniqueness and existence theorems of some basic boundary value problems of the 2D thermoelasticity with microtemperatures are proved and the explicit solutions of boundary value problems for the half-plane are constructed.

In the present paper the linear theory of thermoelasticity with microtemperatures is considered. The representation of regular solution for the equations of steady vibrations of the 3D theory of thermoelasticity with microtemperatures is obtained. We use it for explicitly solving Dirichlet boundary value problem (BVP) of steady vibrations for an elastic space with spherical cavity. The obtained solutions are represented as absolutely and uniformly convergent series.

2. Basic equations

We consider an isotropic elastic material with microtemperatures. Let us assume that D^+ is a ball, of radius R_1, centered at point $O(0,0,0)$ in space E_3 and S is a spherical surface of radius R_1. Denote by D^--whole space with a spherical cavity. $\overline{D^+} := D^+ \bigcup S$, $D^- := E_3 \backslash \overline{D^+}$. Let $\mathbf{x} := (x_1, x_2, x_3) \in E_3$, $\partial \mathbf{x} := \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right)$.

The basic homogeneous system of equations of steady vibrations in the linear theory of thermoelasticity with microtemperatures has the following form [2]

$$\mu \Delta \mathbf{u} + (\lambda + \mu) graddiv \mathbf{u} - \beta grad\theta + \varrho \omega^2 \mathbf{u} = 0 \tag{1}$$

$$k_6 \Delta \mathbf{w} + (k_4 + k_5) graddiv \mathbf{w} - k_3 grad\theta + k_8 \mathbf{w} = 0 \tag{2}$$

$$(k\Delta + a_0)\theta + \beta_0 div \mathbf{u} + k_1 div \mathbf{w} = 0 \tag{3}$$

where $\mathbf{u} = (u_1, u_2)^T$ is the displacement vector, $\mathbf{w} = (w_1, w_2)^T$ is the microtemperature vector, θ is the temperature measured from the constant absolute temperature T_0 $(T_0 > 0)$ by the natural state (i.e. by the state of the absence of loads), $a_0 = i\omega a T_0$, $\beta_0 = i\omega\beta T_0$, $k_8 = ib\omega - k_2$, $b > 0$, a, λ, μ, β, k, k_j, $j = 1, ..., 6$, are constitutive coefficients, Δ is the 3D Laplace operator and ω is the oscillation frequency $(\omega > 0)$. The superscript "T" denotes transposition.

We will suppose that the following assumptions on the constitutive coefficients hold [2]

$$\mu > 0, \quad 3\lambda + 2\mu > 0, \quad a > 0, \quad b > 0, \quad k > 0,$$

$$3k_4 + k_5 + k_6 > 0, \quad k_6 \pm k_5 > 0, \quad (k_1 + k_3 T_0)^2 < 4T_0 k k_2.$$

Definition 1. A vector-function $\mathbf{U}(U_1, U_2, U_3, U_4, U_5, U_6, U_7)$ defined in the domain D^- is called regular if [6]

1.
$$\mathbf{U} \in C^2(D^-) \cap C^1(\overline{D^-}),$$

2.
$$\mathbf{U} = \sum_{j=1}^{5} \mathbf{U}^{(j)}(\mathbf{x}), \quad U^{(j)} = (U_1^{(j)}, U_2^{(j)}, U_3^{(j)}, U_4^{(j)}, U_5^{(j)}, U_6^{(j)}, U_7^{(j)}),$$
$$U^{(j)} \in C^2(D^-) \cap C^1(\overline{D^-}), \tag{4}$$

3.
$$(\Delta + \lambda_j^2) U_l^{(j)} = 0, \tag{5}$$

and

$$\left(\frac{\partial}{\partial |\mathbf{x}|} - i\lambda_j \right) U_l^{(j)} = e^{i\lambda_j |\mathbf{x}|} o(|\mathbf{x}|^{-1}), \quad for \quad |\mathbf{x}| \geq 1, \tag{6}$$

$$U_m^{(5)} = U_7^{(4)} = U_7^{(5)} = 0, \quad m = 1,2,3, \quad j = 1,2,..,5, \quad l = 1,2,...,7,$$

where $\lambda_j^2, \quad j = 1,2,3$ are roots of equation $D(-\xi) = 0$, where

$$D(\Delta) = (\mu_0\Delta + \rho\omega^2)k_1k_3\Delta + (k_7\Delta + k_8)[\beta\beta_0\Delta + (\mu_0\Delta + \rho\omega^2)(k\Delta + a_0)],$$

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = \frac{1}{\mu_0 k k_7}\left[\mu_0(a_0k_7 + kk_8 + k_1k_3) + \rho\omega^2 kk_7 + \beta\beta_0 k_7\right],$$

$$\lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2 = \frac{1}{\mu_0 k k_7}\left[k_8(\mu_0 a_0 + \beta\beta_0) + \rho\omega^2(a_0k_7 + kk_8 + k_1k_3)\right],$$

$$\lambda_1^2\lambda_2^2\lambda_3^2 = \frac{a_0k_8\rho\omega^2}{\mu_0 k k_7} = \frac{a_0\mu k_6\lambda_4^2\lambda_5^2}{\mu_0 k k_7}, \quad \mu_0 = \lambda + 2\mu, \quad k_7 = k_4 + k_5 + k_6,$$

the constants λ_4^2 and λ_5^2 are determined by the formulas

$$\lambda_4^2 = \frac{\rho\omega^2}{\mu} > 0, \quad \lambda_5^2 = \frac{k_8}{k_6}.$$

The quantities $\lambda_j^2, \quad j = 1,2,3,5$ are complex numbers and are chosen so as to ensure positivity of their imaginary part, i.e. it is assumed that $Im\lambda_j^2 > 0$.

Equations in (6) are Sommerfeld-Kupradze type radiation conditions in the linear theory of thermoelastisity with microtemperatures.

The external Dirichlet BVP is formulated as follows:

Find in the unbounded domain D^- a regular solution $\mathbf{U}(\mathbf{u},\mathbf{w},\theta)$ of the equations (1),(2),(3) by the boundary conditions

$$\mathbf{u}^- = \mathbf{F}^-(\mathbf{y}), \quad \mathbf{w}^- = \mathbf{f}^-(\mathbf{y}), \quad \theta^- = f_7^-(\mathbf{y}), \quad \mathbf{y} \in S,$$

where $\mathbf{F}^-(f_1,f_2,f_3)$, $\mathbf{f}^-(f_4,f_5,f_6)$, f_7^- are prescribed functions on S.

The following theorem is valid [6].

Theorem 1. *The external Dirichlet BVP admit at most one regular solution.*

3. Expansion of regular solutions

The following theorem is valid [6].

Theorem 2. *The regular solution $\boldsymbol{U} = (\boldsymbol{u},\boldsymbol{w},\theta) \in C^2(D^-)$ of system (1-3) for $\boldsymbol{x} \in D^-$, is represented as the sum*

$$\mathbf{u} = \sum_{j=1}^{4} \mathbf{u}^{(j)}(\mathbf{x}), \quad \mathbf{w} = \sum_{j=1,2,3,5} \mathbf{w}^{(j)}(\mathbf{x}), \quad \theta = \sum_{j=1}^{3} \theta^{(j)}, \tag{7}$$

where

$$\mathbf{u}^{(j)}=\left[\prod_{l=1;l\neq j}^{4}\frac{\Delta+\lambda_l^2}{\lambda_l^2-\lambda_j^2}\right]\mathbf{u},\quad j=1,2,3,4,$$

$$\mathbf{w}^{(p)}=\left[\prod_{l=1,2,3,5}\frac{\Delta+\lambda_l^2}{\lambda_l^2-\lambda_p^2}\right]\mathbf{w},\quad l\neq p,\quad p=1,2,3,5, \tag{8}$$

$$\theta^{(q)}=\left[\prod_{l=1}^{3}\frac{\Delta+\lambda_l^2}{\lambda_l^2-\lambda_q^2}\right]\theta,\quad l\neq q,\quad q=1,2,3.$$

$\mathbf{u}^{(j)},\mathbf{w}^{(j)}$*and* $\theta^{(j)}$ *are regular functions satisfying the following conditions*

$$(\Delta+\lambda_j^2)\mathbf{u}^{(j)}=0,\quad (\Delta+\lambda_l^2)\mathbf{w}^{(l)}=0,\quad (\Delta+\lambda_m^2)\theta^{(m)}=0,$$

$$j=1,2,3,4,\quad l=1,2,3,5,\quad m=1,2,3.$$

Thus, the regular in D^- solution of system (1-3) is represented as a sum of functions $\mathbf{u}^{(j)}$, $\mathbf{w}^{(j)}$, $\theta^{(j)}$, which satisfy Helmholtz' equations in D^-.

Lemma 1. *In the domain of regularity the regular solution of system (1),(3) can be represented in the form*

$$\mathbf{u}=a_1 grad\varphi_1+a_2 grad\varphi_2+a_3 grad\varphi_3+\mathbf{u}^{(4)},$$

$$\mathbf{w}=b_1 grad\varphi_1+b_2 grad\varphi_2+b_3 grad\varphi_3+\mathbf{w}^{(5)}, \tag{9}$$

$$\theta=\varphi_1+\varphi_2+\varphi_3,$$

where

$$(\Delta+\lambda_j^2)\varphi_j=0,\quad j=1,2,3,\quad (\Delta+\lambda_4^2)\mathbf{u}^{(4)}=0,$$
$$div\mathbf{u}^{(4)}=0,\quad (\Delta+\lambda_5^2)\mathbf{w}^{(5)}=0,\quad div\mathbf{w}^{(5)}=0, \tag{10}$$

a_j *and* b_j, $j=1,2,3$, *are constants.*

Proof. Replacing $\mathbf{u}$, $\mathbf{w}$ and θ by their values from (8), and substituting $\mathbf{u}$, $\mathbf{w}$, θ into (1),(3), after some calculations we obtain

$$(\mu\Delta+\rho\omega^2)(k_7\Delta+k_8)(\mathbf{u}^{(1)}+\mathbf{u}^{(2)}+\mathbf{u}^{(3)})=$$

$$grad\left[-\frac{(\lambda+\mu)k_1k_3}{\beta_0}(\lambda_1^2\varphi_1+\lambda_2^2\varphi_2+\lambda_3^2\varphi_3)+\beta(k_7\Delta+k_8)(\varphi_1+\varphi_2+\varphi_3)\right. \tag{11}$$
$$\left.+\frac{(\lambda+\mu)}{\beta_0}(k\Delta+a_0)(k_7\Delta+k_8)(\varphi_1+\varphi_2+\varphi_3)\right].$$

Equation (11) is satisfied by

$$(\mu\Delta+\rho\omega^2)(k_7\Delta+k_8)\mathbf{u}^{(1)}=$$

$$\left\{\frac{(\lambda+\mu)}{\beta_0}[(a_0-k\lambda_1^2)(k_8-k_7\lambda_1^2)-k_1k_3\lambda_1^2]+\beta(k_8-k_7\lambda_1^2)\right\} grad\varphi_1,$$

$$(\mu\Delta+\rho\omega^2)(k_7\Delta+k_8)\mathbf{u}^{(2)}=$$

$$\left\{\frac{(\lambda+\mu)}{\beta_0}[(a_0-k\lambda_2^2)(k_8-k_7\lambda_2^2)-k_1k_3\lambda_2^2]+\beta(k_8-k_7\lambda_2^2)\right\} grad\varphi_2,$$

$$(\mu\Delta+\rho\omega^2)(k_7\Delta+k_8)\mathbf{u}^{(3)}=$$

$$\left\{\frac{(\lambda+\mu)}{\beta_0}[(a_0-k\lambda_3^2)(k_8-k_7\lambda_3^2)-k_1k_3\lambda_3^2]+\beta(k_8-k_7\lambda_3^2)\right\} grad\varphi_3.$$

last identity gives

$$\mathbf{u}^{(1)}=a_1grad\varphi_1,\quad \mathbf{u}^{(2)}=a_2grad\varphi_2\quad \mathbf{u}^{(3)}=a_3grad\varphi_3 \tag{12}$$

where

$$a_1=\frac{\beta}{\mu\lambda_4^2-\mu_0\lambda_1^2},\quad a_2=\frac{\beta}{\mu\lambda_4^2-\mu_0\lambda_2^2},\quad a_3=\frac{\beta}{\mu\lambda_4^2-\mu_0\lambda_3^2}.$$

Similarly

$$\mathbf{w}^{(1)}=b_1grad\varphi_1,\quad \mathbf{w}^{(2)}=b_2grad\varphi_2\quad \mathbf{w}^{(3)}=b_3grad\varphi_3,$$

where

$$b_1=\frac{k_3}{k_6\lambda_5^2-k_7\lambda_1^2},\quad b_2=\frac{k_3}{k_6\lambda_5^2-k_7\lambda_2^2},\quad b_3=\frac{k_3}{k_6\lambda_5^2-k_7\lambda_3^2}.$$

Thus

$$\begin{aligned}
&\mathbf{u}=a_1grad\varphi_1+a_2grad\varphi_2+a_3grad\varphi_3+\mathbf{u}^{(4)}=\sum_{j=1}^{3}a_jgrad\varphi_j+\mathbf{u}^{(4)},\\
&\mathbf{w}=b_1grad\varphi_1+b_2grad\varphi_2+b_3grad\varphi_3+\mathbf{w}^{(5)}=\sum_{j=1}^{3}b_jgrad\varphi_j+\mathbf{w}^{(5)},\\
&\theta=\varphi_1+\varphi_2+\varphi_3=\sum_{j=1}^{3}\varphi_j,\\
&(\Delta+\lambda_j^2)\varphi_j=0,\quad j=1,2,3,\quad (\Delta+\lambda_4^2)\mathbf{u}^{(4)}=0,\\
&div\mathbf{u}^{(4)}=0,\quad (\Delta+\lambda_5^2)\mathbf{w}^{(5)}=0,\quad div\mathbf{w}^{(5)}=0,
\end{aligned} \tag{13}$$

Now let us prove that if the vector $\mathbf{U}(\mathbf{u},\mathbf{w},\theta)=0$, then $\varphi_1=\varphi_2=\varphi_3=0$, $\mathbf{u}^{(4)}=\mathbf{w}^{(5)}=0$. It follows from (13) that

$$div[a_1grad\varphi_1+a_2grad\varphi_2+a_3grad\varphi_3+\mathbf{u}^{(4)}]=0,$$

$$div[b_1grad\varphi_1+b_2grad\varphi_2+b_3grad\varphi_3+\mathbf{w}^{(5)}]=0,$$

$$\varphi_1(\mathbf{x})+\varphi_2(\mathbf{x})+\varphi_3(\mathbf{x})=0.$$

From these equations we obtain

$$a_1\lambda_1^2\varphi_1 + a_2\lambda_2^2\varphi_2 + a_3\lambda_3^2\varphi_3 = 0,$$

$$b_1\lambda_1^2\varphi_1 + b_2\lambda_2^2\varphi_2 + b_3\lambda_3^2\varphi_3 = 0,$$

$$\varphi_1(\mathbf{x}) + \varphi_2(\mathbf{x}) + \varphi_3(\mathbf{x}) = 0.$$

The determinant of this system is

$$D_1 = \frac{\beta k_3\mu k_6\lambda_4^2\lambda_5^2(\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_3^2)(\lambda_2^2-\lambda_3^2)(k_6\mu_0\lambda_5^2-k_7\mu\lambda_4^2)}{(\rho\omega^2-\mu_0\lambda_1^2)(\rho\omega^2-\mu_0\lambda_2^2)(\rho\omega^2-\mu_0\lambda_3^2)(k_8-k_7\lambda_1^2)(k_8-k_7\lambda_2^2)(k_8-k_7\lambda_3^2)} \neq 0.$$

Thus we have $\varphi_1 = \varphi_2 = \varphi_3 = 0$, $\mathbf{u}^{(4)} = 0$, $\mathbf{w}^{(5)} = 0$ and the proof is completed.

We introduce the notations. If $\mathbf{g}(\mathbf{x}) = \mathbf{g}(g_1, g_2, g_3)$ and $\mathbf{q}(\mathbf{x}) = \mathbf{q}(q_1, q_2, q_3)$, then by symbols $(\mathbf{g.q})$ and $[\mathbf{g.q}]$ will be denoted scalar product and vector product respectively

$$(\mathbf{g.q}) = \sum_{k=1}^{3} g_k q_k, \quad [\mathbf{g.q}] = (g_2q_3 - g_3q_2, g_3q_1 - g_1q_3, g_1q_2 - g_2q_1),$$

Let us consider the metaharmonic equation

$$(\Delta + \nu^2)\psi = 0, \quad Im\nu \neq 0.$$

For this equation the following statements are valid and we cite them without proof.

Lemma 2. *If the regular vector ψ satisfies the conditions*

$$(\Delta + \nu^2)\psi = 0, \quad Im\nu \neq 0, \quad div\psi = 0,$$

$$(\mathbf{x}\cdot\psi) = 0, \quad \mathbf{x} \in D^+(orD^-),$$

then it can be represented in the form

$$\psi(\mathbf{x}) = [\mathbf{x}\cdot\nabla]h(\mathbf{x}),$$

where

$$(\Delta + \nu^2)h(\mathbf{x}) = 0, \quad \nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right).$$

In addition if

$$\int_{S(0,a)} h(\mathbf{x})ds = 0,$$

where $S(0,a) \subset D^+(orD^-)$ is an arbitrary spherical surface of radius a, then between the vector ψ and the function h there exists one-to-one correspondence.

Lemma 3. *If the regular vector ψ satisfies the conditions*

$$(\Delta + \lambda^2)\psi = 0, \quad Im\lambda \neq 0 \quad div\psi = 0, \quad \mathbf{x} \in D^+(orD^-),$$

then it can be represented in the form

$$\psi(\mathbf{x}) = [\mathbf{x}\cdot\nabla]\varphi_3(\mathbf{x}) + rot[\mathbf{x}\cdot\nabla]\varphi_4(\mathbf{x}),$$

where

$$(\Delta + \lambda^2)\varphi_j = 0, \quad j = 3, 4.$$

I*n addition if*

$$\int\limits_{S(0,a)} \varphi_j ds = 0, \quad j = 3, 4,$$

where $S(0, a) \subset D^+(or D^-)$ *is an arbitrary spherical surface of radius* a*, then between the vector* ψ *and the functions* φ_j*,* $j = 1, .., 4$*, there exists one-to-one correspondence.*

Lemma 2 and Lemma 3 are proved in [13].

Lemma 2 and Lemma 3 lead to the following result.

Theorem 3. *The vector* $\boldsymbol{U} = (\boldsymbol{u}, \boldsymbol{w}, \theta)$*, is a regular solution of the homogeneous equations (1),(3), in* $D^+(or D^-)$*, if and only if, when it is represented in the form*

$$\begin{aligned}
&\mathbf{u}(\mathbf{x}) = \sum_{j=1}^{3} a_j \; grad\varphi_j + \frac{\mu}{\rho\omega^2} \; rot\psi^3(\mathbf{x}),\\
&\mathbf{w}(\mathbf{x}) = \sum_{j=1}^{3} b_j grad\varphi_j + \frac{k_6}{k_8} \; rot\varphi^3(\mathbf{x}),\\
&\theta(\mathbf{x}) = \varphi_1(\mathbf{x}) + \varphi_2(\mathbf{x}) + \varphi_3(\mathbf{x}),
\end{aligned} \tag{14}$$

where

$$\begin{aligned}
&(\Delta + \lambda_4^2)\psi^3 = 0, \quad div\psi^3 = 0,\\
&(\Delta + \lambda_5^2)\varphi^3 = 0, \quad div\varphi^3 = 0,\\
&\psi^3(\mathbf{x}) = [\mathbf{x} \cdot \nabla]\psi_3(\mathbf{x}) + rot[\mathbf{x} \cdot \nabla]\psi_4(\mathbf{x}),\\
&\varphi^3(\mathbf{x}) = [\mathbf{x} \cdot \nabla]\varphi_4(\mathbf{x}) + rot[\mathbf{x} \cdot \nabla]\varphi_5(\mathbf{x}),
\end{aligned} \tag{15}$$

$$\int\limits_{S(0,a)} \psi_j ds = 0, \quad (\Delta + \lambda_4^2)\psi_j = 0, \quad j = 3, 4,$$

$$\int\limits_{S(0,a)} \varphi_j ds = 0, \quad (\Delta + \lambda_5^2)\varphi_j = 0, \quad j = 4, 5,$$

$S(0, a) \subset D^+(or D^-)$ *is an arbitrary spherical surface of radius* a*. Between the vector* $\boldsymbol{U}(\boldsymbol{x}) = (\boldsymbol{u}, \boldsymbol{w}, \theta)$ *and the functions* φ_j*,* ψ_j $j = 1, .., 4$*, there exists one-to-one correspondence.*

Remark. By virtue of the equality

$$rotrot[x.\nabla]\varphi_4 = -\Delta[x.\nabla]\varphi_4,$$

formula (14) can be written as

$$\mathbf{u}(\mathbf{x}) = \sum_{j=1}^{3} a_j grad\varphi_j - [\mathbf{x}\cdot\nabla]\psi_4(\mathbf{x}) + \frac{\mu}{\rho\omega^2}\, rot[\mathbf{x}\cdot\nabla]\psi_3(\mathbf{x}),$$

$$\mathbf{w}(\mathbf{x}) = \sum_{j=1}^{3} b_j grad\varphi_j - [\mathbf{x}\cdot\nabla]\varphi_5(\mathbf{x}) + \frac{k_6}{k_8}\, rot[\mathbf{x}\cdot\nabla]\varphi_4(\mathbf{x}), \qquad (16)$$

$$\theta(\mathbf{x}) = \varphi_1(\mathbf{x}) + \varphi_2(\mathbf{x}) + \varphi_3(\mathbf{x}).$$

Below we shall use solution (16) to solve the Dirichlet boundary value problem of steady vibrations for an elastic space with spherical cavity.

4. Some auxiliary formulas

In the sequel we use the following notations: let us introduce the spherical coordinates

$$x_1 = \rho\sin\vartheta\cos\varphi, \quad x_2 = \rho\sin\vartheta\sin\varphi, \quad x_3 = \rho\cos\vartheta,$$

$$y_1 = R_1\sin\vartheta_0\cos\varphi_0, \quad y_2 = R_1\sin\vartheta_0\sin\varphi_0, \quad y_3 = R_1\cos\vartheta_0, \quad y\in S, \qquad (17)$$

$$\rho^2 = x_1^2 + x_2^2 + x_3^2, \quad 0\le\vartheta\le\pi, \quad 0\le\varphi\le 2\pi \quad 0\le\rho\le R_1.$$

The operator $\dfrac{\partial}{\partial S_k(\mathbf{x})}$ is determined as follows

$$[\mathbf{x}\cdot\nabla]_k = \frac{\partial}{\partial S_k(\mathbf{x})} \quad k=1,2,3 \quad \mathbf{x}\in E_3,$$

Simple calculations give

$$\frac{\partial}{\partial S_1(\mathbf{x})} = x_2\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_2} = -cos\varphi ctg\vartheta\frac{\partial}{\partial\varphi} - sin\varphi\frac{\partial}{\partial\vartheta},$$

$$\frac{\partial}{\partial S_2(\mathbf{x})} = x_3\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_3} = -sin\varphi ctg\vartheta\frac{\partial}{\partial\varphi} + cos\varphi\frac{\partial}{\partial\vartheta},$$

$$\frac{\partial}{\partial S_3(\mathbf{x})} = x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1} = \frac{\partial}{\partial\varphi}.$$

The following identities are true [13]

$$(\mathbf{x}\cdot rotg(\mathbf{x})) = \sum_{k=0}^{3}\frac{\partial g_k(\mathbf{x})}{\partial S_k(\mathbf{x})}, \quad \sum_{k=0}^{3}\frac{\partial}{\partial S_k(\mathbf{x})}(rot[\mathbf{x}\cdot\nabla]h)_k = 0,$$

$$\sum_{k=0}^{3}\frac{\partial}{\partial S_k(\mathbf{x})}(rot\mathbf{g}(\mathbf{x}))_k = \rho\frac{\partial}{\partial\rho}div\mathbf{g}(\mathbf{x}) - \sum_{k=0}^{3} x_k\Delta\mathbf{g}_k(\mathbf{x}),$$

$$\sum_{k=0}^{3}\frac{\partial}{\partial S_k(\mathbf{x})}[\mathbf{x}\cdot\mathbf{g}]_k=\rho^2 div\mathbf{g}(\mathbf{x})-(\mathbf{x}\cdot\mathbf{g}(\mathbf{x}))-\rho\frac{\partial}{\partial\rho}(\mathbf{x}\cdot\mathbf{g}(\mathbf{x})),$$

$$\sum_{k=1}^{3}\frac{\partial}{\partial S_k(\mathbf{x})}[\mathbf{x}\cdot rot\mathbf{g}(\mathbf{x})]_k=-(\rho\frac{\partial}{\partial\rho}+1)\sum_{k=0}^{3}\frac{\partial g_k(\mathbf{x})}{\partial S_k(\mathbf{x})},$$

$$\sum_{k=0}^{3}x_k\frac{\partial}{\partial S_k(\mathbf{x})}=0,\quad \frac{\partial}{\partial S_k(\mathbf{x})}\frac{\partial}{\partial x_k}=\frac{\partial}{\partial x_k}\frac{\partial}{\partial S_k(\mathbf{x})},\tag{18}$$

$$\sum_{k=0}^{3}\frac{\partial^2}{\partial S_k^2(\mathbf{x})}=\frac{\partial^2}{\partial\vartheta^2}+ctg\vartheta\frac{\partial}{\partial\vartheta}+\frac{1}{sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2},\quad \frac{\partial x_k}{\partial S_k}=0,$$

$$\sum_{k=0}^{3}\frac{\partial}{\partial S_k(\mathbf{x})}\frac{\partial}{\partial x_k}=0,\quad \frac{\partial g(\rho)Y(\vartheta,\varphi)}{\partial S_k(\mathbf{x})}=g(\rho)\frac{\partial Y(\vartheta,\varphi)}{\partial S_k(\mathbf{x})}.$$

Let

$$(\mathbf{z}\cdot\mathbf{F}^-)=h_1^-(\mathbf{z}),\qquad \sum_{k=1}^{3}\frac{\partial}{\partial S_k(\mathbf{z})}[\mathbf{z}\cdot\mathbf{F}^-]_k=h_2^-(\mathbf{z}),\qquad \sum_{k=1}^{3}\frac{\partial}{\partial S_k(\mathbf{z})}F_k^-=h_3^-(\mathbf{z}),$$

$$(\mathbf{z}\cdot\mathbf{f}^-)=h_4^-(\mathbf{z}),\qquad \sum_{k=1}^{3}\frac{\partial}{\partial S_k(\mathbf{z})}[\mathbf{z}\cdot\mathbf{f}^-]_k=h_5^-(\mathbf{z}),\quad \sum_{k=1}^{3}\frac{\partial}{\partial S_k(\mathbf{z})}f_k^-=h_6^-(\mathbf{z}),\quad f_7^-=h_7^-(\mathbf{z}).$$

Let us assume that f_k. $k=1,..,7$ are sufficiently smooth(differentiable) functions. Let us expand the functions h_k in spherical harmonics

$$h_k^-(\mathbf{z})=\sum_{m=0}^{\infty}h_{km}^-(\vartheta,\varphi),$$

where h_{km}^- is the spherical harmonic of order m :

$$h_{km}^-=\frac{2m+1}{4\pi R_1^2}\int_S P_m(\cos\gamma)h_k^-(\mathbf{y})dS_y,$$

P_m is Legendre polynomial of the m-th order, γ is an angle formed by the radius-vectors Ox and Oy,

$$\cos\gamma=\frac{1}{|\mathbf{x}||\mathbf{y}|}\sum_{m=1}^{3}x_k y_k.$$

From these formulas it follows that if g_m is the spherical harmonic the operator $\dfrac{\partial}{\partial S_k}$, $k=1,2,3$, does not affect the order of the spherical function:

$$\sum_{k=0}^{3}\frac{\partial^2 g_m(\mathbf{x})}{\partial S_k^2(\mathbf{x})}=-m(m+1)g_m(\mathbf{x}).$$

The general solutions of the equations $(\Delta+\lambda_k^2)\psi=0, \quad k=1,2,3,4,5,$ in the domain D^- have the form [13]

$$\psi(x)=\sum_{m=0}^{\infty}\Psi_m^{(1)}(\lambda_k\rho)Y_m(\vartheta,\varphi), \quad \rho>R_1, \tag{19}$$

where

$$\Psi_m^{(1)}(\lambda_k\rho)=\frac{\sqrt{R_1}H^{(1)}_{m+\frac{1}{2}}(\lambda_k\rho)}{\sqrt{\rho}H^{(1)}_{m+\frac{1}{2}}(\lambda_k R_1)}.$$

5. The Dirichlet BVP for an infinite space with the spherical cavity

The solution of the Dirichlet BVP problem

$$\mathbf{u}^-=\mathbf{F}^-(f_1,f_2,f_3), \quad \mathbf{w}^-=\mathbf{f}^-(f_4,f_5,f_6), \quad \theta^-=f_7^-$$

in the domain D^- is sought in the form (16).

From (16) we get

$$\begin{aligned}
&(\mathbf{x}\cdot\mathbf{u})=\sum_{k=1}^{3}a_k\rho\frac{\partial\varphi_k}{\partial\rho}+c_1\sum_{k=1}^{3}\frac{\partial^2\psi_3}{\partial S_k^2(\mathbf{x})},\\
&\sum_{k=1}^{3}\frac{\partial}{\partial S_k(\mathbf{x})}[\mathbf{x}\cdot\mathbf{u}]_k=a_1\sum_{k=1}^{3}\frac{\partial^2\varphi_1}{\partial S_k^2(\mathbf{x})}+a_2\sum_{k=1}^{3}\frac{\partial^2\varphi_2}{\partial S_k^2(\mathbf{x})}\\
&+a_3\sum_{k=1}^{3}\frac{\partial^2\varphi_3}{\partial S_k^2(\mathbf{x})}-c_1(\rho\frac{\partial}{\partial\rho}+1)\sum_{k=1}^{3}\frac{\partial^2\psi_3}{\partial S_k^2(\mathbf{x})},\\
&\sum_{k=1}^{3}\frac{\partial u_k}{\partial S_k(\mathbf{x})}=\sum_{k=1}^{3}\frac{\partial^2\psi_4}{\partial S_k^2(\mathbf{x})}, \quad (\mathbf{x}\cdot\mathbf{w})=\sum_{k=1}^{3}b_k\rho\frac{\partial\varphi_k}{\partial\rho}+c_2\sum_{k=1}^{3}\frac{\partial^2\varphi_4}{\partial S_k^2(\mathbf{x})},\\
&\sum_{k=1}^{3}\frac{\partial}{\partial S_k(\mathbf{x})}[\mathbf{x}\cdot\mathbf{w}]_k=b_1\sum_{k=1}^{3}\frac{\partial^2\varphi_1}{\partial S_k^2(\mathbf{x})}+b_2\sum_{k=1}^{3}\frac{\partial^2\varphi_2}{\partial S_k^2(\mathbf{x})}\\
&+b_3\sum_{k=1}^{3}\frac{\partial^2\varphi_3}{\partial S_k^2(\mathbf{x})}-c_2(\rho\frac{\partial}{\partial\rho}+1)\sum_{k=1}^{3}\frac{\partial^2\varphi_4}{\partial S_k^2(\mathbf{x})},\\
&\sum_{k=1}^{3}\frac{\partial w_k}{\partial S_k(\mathbf{x})}=\sum_{k=1}^{3}\frac{\partial^2\varphi_5}{\partial S_k^2(\mathbf{x})}, \quad \theta=\sum_{k=1}^{3}\varphi_k, \quad c_1=\frac{1}{\lambda_4^2}, \quad c_2=\frac{1}{\lambda_5^2}.
\end{aligned}\tag{20}$$

Suppose the functions $\varphi_m(\mathbf{x}), \quad m=1,2,3,4,5, \quad and \quad \psi_j, \quad j=3,4,$ are sought in the form

$$\varphi_k(\mathbf{x}) = \sum_{m=0}^{\infty} \Psi_m^{(1)}(\lambda_k\rho) Y_{km}(\vartheta,\varphi), \quad k = 1,2,3,$$

$$\varphi_j(\mathbf{x}) = \sum_{m=0}^{\infty} \Psi_m^{(1)}(\lambda_5\rho) Y_{jm}(\vartheta,\varphi), \quad j = 4,5, \tag{21}$$

$$\psi_j(\mathbf{x}) = \sum_{m=0}^{\infty} \Psi_m^{(1)}(\lambda_4\rho) Z_{jm}(\vartheta,\varphi), \quad j = 3,4, \quad \rho > R_1,$$

where Y_{km}, *and* Z_{jm} are the unknown spherical harmonic of order m,

$$\Psi_m^{(1)}(\lambda_k\rho) = \frac{\sqrt{R_1} H^{(1)}_{m+\frac{1}{2}}(\lambda_k\rho)}{\sqrt{\rho} H^{(1)}_{m+\frac{1}{2}}(\lambda_k R_1)}.$$

Remark. The conditions $\int\limits_{S(0,a)} \psi_j ds = 0, \quad j = 3,4, \int\limits_{S(0,a)} \varphi_j ds = 0, \quad j = 4,5$ in fact mean that

$$Y_{40} = Y_{50} = Z_{30} = Z_{40} = 0.$$

Substituting the expressions of $\varphi_m(x), \quad m = 1,2,3,4,5$ and $\psi_j(x), \quad j = 3,4$ in (20), we obtain

$$(\mathbf{x}\cdot\mathbf{u}) = \sum_{k=1}^{3}\sum_{m=0}^{\infty} a_k \rho \frac{\partial}{\partial\rho} \Psi_m^{(1)}(\lambda_k\rho) Y_{km} - c_1 \sum_{m=0}^{\infty} m(m+1)\Psi_m^{(1)}(\lambda_4\rho) Z_{3m},$$

$$\sum_{k=1}^{3} \frac{\partial}{\partial S_k(\mathbf{x})}[\mathbf{x}\cdot\mathbf{u}]_k =$$

$$\sum_{m=0}^{\infty} m(m+1)\left\{-\sum_{k=1}^{3} a_k \Psi_m^{(1)}(\lambda_k\rho) Y_{km} + c_1(\rho\frac{\partial}{\partial\rho}+1)\Psi_m^{(1)}(\lambda_4\rho) Z_{3m},\right\},$$

$$\sum_{k=1}^{3} \frac{\partial u_k}{\partial S_k(\mathbf{x})} = -\sum_{m=0}^{\infty} m(m+1)\Psi_m^{(1)}(\lambda_4\rho) Z_{4m}, \tag{22}$$

$$(\mathbf{x}\cdot\mathbf{w}) = \sum_{k=1}^{3}\sum_{m=0}^{\infty} b_k \rho \frac{\partial}{\partial\rho} \Psi_m^{(1)}(\lambda_k\rho) Y_{km} - c_2 \sum_{m=0}^{\infty} m(m+1)\Psi_m^{(1)}(\lambda_5\rho) Y_{4m},$$

$$\sum_{k=1}^{3} \frac{\partial}{\partial S_k(\mathbf{x})}[\mathbf{x}\cdot\mathbf{w}]_k =$$
$$\sum_{m=0}^{\infty} m(m+1)\left\{-\sum_{k=1}^{3} b_k \Psi_m^{(1)}(\lambda_k\rho) Y_{km} + c_2(\rho\frac{\partial}{\partial\rho}+1)\Psi_m^{(1)}(\lambda_5\rho) Y_{4m},\right\},$$

$$\sum_{k=1}^{3} \frac{\partial w_k}{\partial S_k(\mathbf{x})} = -\sum_{m=0}^{\infty} m(m+1)\Psi_m^{(1)}(\lambda_5\rho) Y_{5m}, \quad \theta = \sum_{k=1}^{3}\sum_{m=0}^{\infty} \Psi_m^{(1)}(\lambda_k\rho) Y_{km}(\vartheta,\varphi).$$

Passing to the limit as $\rho \to R_1$ and taking into account boundary conditions for the determination of Y_{mj} and Z_{mj} we obtain the system of algebraic equations

$$\sum_{k=1}^{3} a_k \left[\rho \frac{\partial}{\partial \rho} \Psi_m^{(1)}(\lambda_k \rho)\right]_{\rho=R_1} Y_{km} - c_1 m(m+1) Z_{3m} = h_{1m}^{-},$$

$$m(m+1)\left\{-\sum_{k=1}^{3} a_k Y_{km} + c_1 \left[(\rho\frac{\partial}{\partial\rho}+1)\Psi_m^{(1)}(\lambda_4\rho)\right]_{\rho=R_1} Z_{3m}\right\} = h_{2m}^{-},$$

$$-m(m+1)Z_{4m} = h_{3m}^{-},$$

$$\sum_{k=1}^{3} b_k \left[\rho \frac{\partial}{\partial \rho} \Psi_m^{(1)}(\lambda_k \rho)\right]_{\rho=R_1} Y_{km} - c_2 m(m+1) Y_{4m} = h_{4m}^{-},$$

$$m(m+1)\left\{-\sum_{k=1}^{3} b_k Y_{km} + c_2 \left[(\rho\frac{\partial}{\partial\rho}+1)\Psi_m^{(1)}(\lambda_5\rho)\right]_{\rho=R_1} Y_{4m},\right\} = h_{5m}^{-},$$

$$-m(m+1)Y_{5m} = h_{6m}^{-}, \quad Z_{40} = Y_{40} = Z_{30} = Y_{50} = 0,$$

$$Y_{1m} + Y_{2m} + Y_{3m} = h_{7m}^{-}, \quad h_{30}^{-} = h_{60}^{-} = h_{20}^{-} = h_{50}^{-} = 0. \tag{23}$$

By virtue of Theorem 1 we conclude that the system (23) for $m \geq 0$ is uniquely solvable and the functions Y_{jm} and Z_{jm} are possible to express by the known functions h_{jm}^{-}.

If we take into account the sufficient conditions of convergence of absolutely and uniformly convergent series with respect to the spherical harmonic and the property of functions $\Psi_m^{(1)}(\lambda_k\rho)$ we conclude that the obtained solutions are represented as absolutely and uniformly convergent series.

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