NEW SUFFICIENT CONDITION FOR UNIFORM INTEGRABILITY OF STOCHASTIC EXPONENTIAL

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Given a basic probability space $(\Omega, \mathcal{F}, P)$ with continuous filtration $(\mathcal{F}_t)_{0\le t\le T}$ and let $M = (M_t)_{0\le t\le T}$ be a local martingale. Denote by $\mathcal{E}(M)$ the stochastic exponential of a local martingale M: $\mathcal{E}_t(M) = \exp\left(M_t - \frac{1}{2}\langle M\rangle_t\right)$.. Now let us formulate the main result of the paper:

Theorem: Let a_s be a predictable process such that $|a_s - 1| \ge \varepsilon$ for some $\varepsilon > 0$ and

$$\sup_{0\le v\le T} E\exp\left\{\int_0^v a_s dM_s + \int_0^v \left(\frac{1}{2} - a_s\right) d\langle M\rangle_s\right\} < \infty$$

where sup is taken over all stopping times. Then for any predictable process b_s such that $|b_s - a_s| \le |a_s - 1|$ the stochastic exponential $\mathcal{E}(\int bdM)$ is a uniformly integrable martingale.

Notice that the inequality $|b_s - a_s| \le |a_s - 1|$ is obviously satisfied when $b_s \equiv 1$ and $b_s = a_s$, so we will have the following important corollary which is the generalization of mixed Novikov-Kazamaki's criteria:

Corollary: If conditions of theorem are satisfied, then $\mathcal{E}(M)$ and $\mathcal{E}(\int adM)$ are uniformly integrable martingales.

Notice that Novikov's, Kazamaki's and mixed Novikov-Kazamaki's criteria correspond to the case where $a_s \equiv 0$, $a_s \equiv \frac{1}{2}$ *and* $a_s \equiv a \neq 1$ *is a constant, respectively.*

Now we sketch the main idea of the proof. At first let us define the stopping time

$\tau = \inf\{t \ge 0 : \mathcal{E}_t(\int bdM) = 0\}$ and the process:

$$Y_t = E\left[\mathcal{E}_{t,T}\left(\int bdM\right) I_{(t<\tau)} + I_{(t\ge\tau)} | \mathcal{F}_t\right]$$

where $\mathcal{E}_{t,T}(\int bdM) = \mathcal{E}_T(\int bdM)/\mathcal{E}_t(\int bdM)$. Since$\mathcal{E}(\int bdM)$ is a nonnegative super martingale $\mathcal{E}_t(\int bdM) = 0$ if $t \ge \tau$, and from this we can deduce that $Y_t\mathcal{E}_t(\int bdM)$ is also a super martingale. After that we will derive corresponding backward equation on the stochastic interval $[[0;\tau[[$ for the process Y_t and then using technique of backward stochastic differential equations we prove the equality $Y_0 = E\mathcal{E}_T(\int bdM) = 1$, which means that $\mathcal{E}(\int bdM)$ is a uniformly integrable martingale.