Reports of Enlarged Sessions of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 34, 2020

TO THE CREATION OF CONSISTENT MODELS FOR OF THIN-WALLED STRUCTURES

Tamaz Vashakmadze

Abstract. This report represents the part of works, dedicated to the creation of consistent 2D boundary value problems corresponding, to elastic thin-walled structures (TWS), an analysis for Kármán type system of DEs without variety of ad hoc assumptions, since in the classical form of this system, one of them represents the condition of compatibility. Then we find the general solution of nonlinear systems by development methodology of generalized analysis functions theory for some class of complex systems of DEs, containing the integrals both of Volterra and Fredholm type.

Keywords and phrases: Elastic thin-walled structures, generalized analytic functions theory, Monge-Ampére operator, Volterra and Fredholm type nonlinear pseudo-integral equations.

AMS subject classification (2010): 30G20, 74J40, 74k20.

1 Introduction. Let us consider the equilibrium equations of the elastic body in the form [1, 2]:

$$\partial_j(\sigma_{ij} + \sigma_{kj}u_{i,k}) = f_i, \quad x \in \Omega_h = D(x,y) \times \left]h^-(x,y),\ h^+(x,y)\right[, \tag{1}$$

with the boundary conditions:

$$T_{i3} = \sigma_{i3} + \sigma_{j3}u_{i,j} = g_i^{\pm}, \quad x \in S^{\pm} = D \times \{h^{\pm}\}, \quad T_3 = (T_{13},\ T_{23},\ T_{33})^T, \tag{2}$$

$$l\,[\partial_1,\ \partial_2,\ \partial_3]\,(x,u) = g, \quad x \in S = \partial D \times \left]h^-,\ h^+\right[. \tag{3}$$

The relation between the displacement vector $u = (u_1,\ u_2,\ u_3)$, the symmetrical strain ε and stress σ tensors satisfy the Cauchy formulae and Hooke's law:

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{i,k}u_{j,k}), \ \ \varepsilon = A\sigma, \ \ \sigma = B\varepsilon, \ \ i,j = 1,2,3. \tag{4}$$

2 On problems of constructing von Kármán type systems. Among the works, dedicated to the construction and justification of the plate and shell theory, a special mention should be made to the monograph [1]. In particular, Ciarlet wrote: "The 2D of von Kármán equations for nonlinearly elastic plates play an almost mythical role in applied mathematics. While they have been abundantly, and satisfactorily, studied from the mathematical standpoint, as regards notably various questions of existence, regularity, and bifurcation of their solutions, their physical soundness has often been seriously questioned".Based on works [2 - 4] the method of constructing the anisotropic inhomogeneous 2D nonlinear models of von Kármán-Mindlin-Reissner (KMR) type for binary mixture of porous, piezo-magneto-electric and electrically conductive and viscous elastic TWS with variable thickness is given. In particular, the problem of "Physical Soundness" for von Kármán system was solved fully. Against elaborations [1, Ch. 5] the corresponding variables are the quantities with physical meaning such as the averaged components of the displacement vector, bending and twisting moments, shearing forces, rotation of normals, surface efforts. For isotropic and generalized transversal elastic plates in the linear case KMR have the unified representation as the systems of Cauchy-Riemann DEs in terms of planar expansion and rotation. Below for clearness and simplicity consider static problems of the theory of elasticity. Then for an isotropic case we have:

$$\begin{aligned}
&D\Delta^2 u_3^* = \left(1-\tfrac{h^2(1+2\gamma)(2-\nu)}{3(1-\nu)}\Delta\right)\left(g_3^+-g_3^-\right)+2h\left(1-\tfrac{2h^2(1+2\gamma)}{3(1-\nu)}\Delta\right)L\left[u_3^*,F_*\right]\\
&+\left(g_{\alpha,\alpha}^+ + g_{\alpha,\alpha}^-\right)-\int_{-h}^{h}\left(tf_{\alpha,\alpha}-\left(1-\tfrac{1}{1-\nu}(h^2-t^2)\Delta\right)f_3\right)dt+R_8\left[u_3^*;\gamma\right],\\
&Q_{\alpha 3}-\tfrac{1+2\gamma}{3}h^2\Delta Q_{\alpha 3}=-D\Delta u_{3,\alpha}^*+\tfrac{h^2(1+2\gamma)}{3(1-\nu)}\partial_\alpha\left(g_3^+-g_3^-+2h(1+\nu)L\left[u_3^*,F_*\right]\right)\\
&+h\left(g_\alpha^+ + g_\alpha^-\right)-\int_{-h}^{h}\left(tf_\alpha-\tfrac{1+\nu}{2(1-\nu)}(h^2-t^2)f_{3,\alpha}\right)dt+R_{5+\alpha}\left[Q_{\alpha 3};\gamma\right].
\end{aligned}\tag{5}$$

As it is known, even in the case of an isotropic elastic plate of constant thickness the subject of justification was an unsolved problem. The point is that von Kármán, Love, Timoshenko, L. Landau, Lukasiewicz, Washizu,... considered Saint-Venant-Beltrami compatibility condition as one of the equations of the corresponding system of DEs. In [3] we have proved that all DEs systems of von KMR type follow from (1).

We have the following relation (decomposition of Monge-Ampére operator):

$$\begin{aligned}
&L[u,v]=[u,v]=\partial_1[\partial_1(\partial_2 u\partial_2 v)-\partial_2(\partial_1 u\partial_2 v)]-\partial_2[\partial_2(\partial_1 u\partial_2 v)-\partial_1(\partial_2 u\partial_1 v)]\\
&=-(\partial_{11}u\partial_{22}v-2\partial_{12}u\partial_{12}v+\partial_{22}u\partial_{11}v).
\end{aligned}\tag{$M-A$}$$

It is necessary that to system (5) we must add, for evidence, part of von Kármán type system (an isotropic case, see [3, formula (17)]:

$$\begin{aligned}
&(\lambda^*+2\mu)\partial_1\tau+\mu\partial_2\omega=\frac{1}{2h}\bar f_1+\mu(\partial_1(\bar u_{3,2})^2-\partial_2(\bar u_{3,1}\bar u_{3,2}))+\lambda_1(\sigma_{33,1},1)+R_4,\\
&(\lambda^*+2\mu)\partial_2\tau-\mu\partial_1\omega=\frac{1}{2h}\bar f_2+\mu(\partial_2(\bar u_{3,1})^2-\partial_1(\bar u_{3,1}\bar u_{3,2}))+\lambda_1(\sigma_{33,2},1)+R_5.
\end{aligned}\tag{6}$$

Here $\tau=\bar\varepsilon_{\alpha\alpha}$, $\omega=\bar u_{1,2}-\bar u_{2,1}$ are plane expansion and rotation, $\lambda_1=\lambda/2h(\lambda+2\mu)$, nonlinear terms represent a decomposition of Monge-Ampére operator if in $(M-A)$, $u=v=u_3$.

3 On applications of the complex variable function theory. The representations (5) allow to apply complex analysis. Let us preliminarily consider the first equation

(5) and underline the main members:

$$D'\Delta[u,\varphi] = D'([\Delta u,\varphi] + [u,\Delta\varphi] + 2[\partial_\alpha u, \partial_\alpha \gamma]), \quad D\Delta^2 u. \tag{7}$$

The calculate and analysis of these expressions (7) of a symbolical determinant show that the characteristic forms of systems type (5) may be positive, negative or zero values as they represent arbitrary functions of $x,\ y$.

We form the following iterative-direct (hybrid) method for finding the solution of rewriting in complex variables systems of PDEs (5), (6).

Let $[U(z,\bar z)]^{[m]}$ denote $m-th$ approach for deflection $u_3^*(x,y)$ which is calculated by the known right-hand terms without R and $(m-1)$-th order approach of summand

$$\tfrac{2Eh}{16D}\left(1-\tfrac{h^2(1+2\gamma)}{3(1-\nu)}\partial_{\bar z}\partial_z\right)\int_0^z\int_0^{\bar z}(z-\zeta)(\bar z-\bar\zeta)[U,V]^{[m-1]}d\zeta d\bar\zeta, \quad EV=\Phi\left(\tfrac{z+\bar z}{2},\tfrac{z-\bar z}{2i}\right), \tag{8}$$

we do some operations for the second equation of (5) for shearing forces and for system (6). This system is equivalent to the following equation (see [3]):

$$\Delta(\sigma_{11}+\sigma_{22}) = -\frac{E}{2}[w,w] + \frac{\nu}{2h}\int_{-h}^{h}\Delta\sigma_{33}dt + \frac{1+\nu}{2h}\bar f_{\alpha,\alpha}, \quad (K-R)_2.$$

Let

$$V^{[m]} = V^{[m]}(z,\bar z) = -\tfrac{\mu}{\lambda^*+2\mu}\int_0^{\bar z}\int_0^{z}(\bar z-\bar\zeta)(z-\zeta)[U^{[m-1]},U^{[m-1]}]d\vec{\zeta}d\zeta + F(\bar z,z). \tag{9}$$

Thus, by means of complex analysis we must investigate (8). An iterative scheme, described by (9), corresponds to the solution of Volterra nonlinear integral equations with the contraction operator whereas the processes by schemes (8) contain both Volterra and Fredholm type operators with an arbitrary parameter. The convergence for only Volterra type process is evident. When $\gamma \neq -0.5$ the convergence depends on the Fredholm type operator

$$F_r(U,V) = \partial_{\bar z}\partial_z\lambda\int_0^{\bar z}\int_0^{z}(\bar z-\bar\zeta)(z-\zeta)[U(\zeta,\bar\zeta)V(\zeta,\bar\zeta)]d\bar\zeta d\zeta$$

with an arbitrary parameter, denoted for simplicity by λ. The operator $\lambda^{-1}F(U,V)$ depends on the behavior of expression which may generate different kinds of waves (shock, soliton) functions too and in the cases when they are uniformly bounded functions the process, corresponding to applications of the Fredholm operator will be convergent as the corresponding operator will be a contracted one. More convenient may be Seidel's type iterative scheme: let the initial value be $U^{[0]} = \dfrac{1}{4}z^2\bar z^2$. Then in expressions of type (8) we used $V^{[1]}$ defining from (9) and so on. The following theorem is true

Theorem. *Let us consider the following iterative process:*

$$V^{[m]}(z,\bar z) = a\int_0^z\int_0^{\bar z}(z-\zeta)(\bar z-\bar\zeta)\left[U^{[m-1]},U^{[m-1]}\right]d\zeta d\bar\zeta, \quad m=1,2,...,$$

$$U^{[m]}(z,\bar z) = b\int_0^z\int_0^{\bar z}(z-\zeta)(\bar z-\bar\zeta)\left[U^{[m-1]},V^{[m]}\right]d\zeta d\bar\zeta$$

$$+c\int_0^z\int_0^{\bar z}\left[U^{[m-1]},V^{[m]}\right]d\zeta d\bar\zeta, \quad m=1,2,...,$$

then it is convergence for all finite $a,\ b,\ |c| < \frac{4}{3}$, $U^{[0]} = z^n \bar{z}^n$ *and an integer* $\forall n \geq 2$.

Proof. The essential moment is estimation of the transition effect from m step to $m+1$ step. Let $U^{[m]} = z^p \bar{z}^p$. The transition process contains two stages: the calculation of expressions of the type $[u, v]$ and corresponding integrals. It is evident that

$$\left[U^{[m]}, U^{[m]}\right] = 2(p(p-1))^2 z^{2p-2} z^{2p-2} - 2p^4 z^{2p-2} \bar{z}^{2p-2} = -2p^2(2p-1) z^{2p-2} \bar{z}^{2p-2},$$

then we also have:

$$V^{[m+1]} = -\frac{2ap^2(2p-1)}{4p^2(2p-1)^2} z^{2p} \bar{z}^{2p} = -\frac{a}{2(2p-1)} z^{2p} \bar{z}^{2p},$$

and

$$\left[U^{[m]}, V^{[m+1]}\right] = -\frac{2ap^2(3p-1)}{2p-1} z^{2p-2} \bar{z}^{2p-2}, \quad c_p = \frac{2p^2(3p-1)}{2p-1},$$

$$I_1 = abc_p \int_0^z \int_0^{\bar{z}} (z-\zeta)(\bar{z}-\bar{\zeta}) z^{2p-2} \bar{z}^{2p-2} dz d\bar{z} = \frac{abc_p}{4p^2(2p-1)^2} z^{2p} \bar{z}^{2p},$$

$$I_2 = acc_p \int_0^z \int_0^{\bar{z}} z^{2p-2} \bar{z}^{2p-2} dz d\bar{z} = \frac{acc_p}{(2p-1)^2} z^{2p-1} \bar{z}^{2p-1},$$

$$c_p (2p-1)^{-2} < \frac{3}{4} + \frac{7}{8(p-1,5)}.$$

This relation shows that if $c = \gamma,\ |\gamma| < \frac{4}{3}$ for all bounded functions a, b the above iterative process is convergence. □

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Received 31.05.2020; accepted 20.09.2020.

Author(s) address(es):

Tamaz Vashakmadze
I. Vekua Institute of Applied Mathematics & Department of Mathematics
of I. Javakhishvili Tbilisi State University
University str. 2, 0186 Tbilisi, Georgia
E-mail: tamazvashakmadze@gmail.com