Zero sets of solutions of the Darboux equation on hyperbolic plane

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Let $\mathcal{L}$ be the Laplace-Beltrami operator on a Riemannian manifold X. The partial differential equation $\mathcal{L}_x(f(x,y)) = \mathcal{L}_y(f(x,y))$ with $f = f(x,y) \in C^2(X \times X)$ is called the generalized Darboux equation. Such equations are of considerable interest in their own right, but they are also important for many applications in geometric analysis and integral geometry (see [1]). In particular, equations of Darboux type are closely connected with the mean value operators on symmetric spaces.

Here we present a new uniqueness theorem for solutions of the generalized Darboux equation for the case where X is a hyperbolic plane.

We take X as the disk $D = \{z \in C : |z| < 1\}$ with the Riemannian structure $ds^2 = \frac{|dz|^2}{(1-|z|^2)^2}$. The Laplace-Beltrami operator for X is given by $L = 4i(1-|z|^2)^2 \frac{\partial^2}{\partial z\, \partial \overline{z}}$. Hence, our equation has the form

$$(1-|z|^2)^2 \frac{\partial^2 f}{\partial z\, \partial \overline{z}} = (1-|w|^2)^2 \frac{\partial^2 f}{\partial w\, \partial \overline{w}}, \tag{1}$$

where $f = f(z,w) \in C^2(D \times D)$.

Theorem 1. *Let $f \in C^2(D \times D)$ satisfy (1). Suppose that $r \in (0,1)$ is given and the following conditions hold.*

(i) $f(z,w) = f(z,|w|)$ *for all* $z, w \in D$.

(ii) $f(z,0) = 0$ *for all* $z \in D$, $|z| \le r$.

(iii) $f(z,w) = 0$ *for all* $z, w \in D$, $|z| = r$.

Then $f = 0$.

We need to say a word about condition (i). It is a familiar fact that if $f(z,w)$ is a radial function of w and $f(z,w) = h(z,\rho)$, $\rho = \operatorname{arth}|w|$, then equation (1) can be rewritten as $4i(1-|z|^2)^2 \frac{\partial^2 h}{\partial z\, \partial \overline{z}} = \frac{\partial^2 h}{\partial \rho^2} + 2\operatorname{cth} 2\rho \frac{\partial h}{\partial \rho}$. This relation is called the non-Euclidean Darboux equation. Some Euclidean analogs of Theorem 1 can be found in [1, Part 5].

References

1. Volchkov, V.V. *Integral Geometry and Convolution Equations.* Kluwer Academic, Dordrecht, 2003.