Laplace-Beltrami Equation on Hypersurfaces and Γ-convergence

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We investigate a mixed boundary value problem for the stationary heat transfer equation in a thin layer $\Omega_h := \mathcal{C} \times [-h, h]$ in $\mathbb{R}^3$ of the thickness $2h$, where $\mathcal{C}$ is a hypersurface with the boundary $\partial\mathcal{C}$. Namely, we consider the BVP Let us consider the mixed BVP with zero Dirichlet but non-zero Neumann data:

$$\begin{aligned} &\Delta_{\Omega_h} T(\omega, t) = f(\omega, t), && (\omega, t) \in \Omega_h, \\ &T^+(\omega, t) = 0, && (\omega, t) \in \partial\Omega_h, \\ &\pm(\partial_t T)^+(\omega, \pm h) = q(\omega, \pm h), && \omega \in \mathcal{C}, \end{aligned} \tag{1}$$

where $\pm\partial_t = \partial_\nu$ represents the normal derivative on the upper and lower boundary surfaces $\mathcal{C} \times \pm h$.

The main object is to find out what happens in Γ-limit when the thickness of the layer $2h$ converges to zero and how the limit is related to the Dirichlet boundary value problem for the Laplace-Beltrami equation on the surface $\mathcal{C}$:

$$\begin{aligned} &\Delta_{\mathcal{C}} T(\omega) = f^0(\omega) + q^0(\omega) \quad \omega \in \mathcal{C}, \\ &T^+(\omega) = 0, \qquad \omega \in \partial\mathcal{C}. \end{aligned} \tag{2}$$

Here

$$\Delta_{\mathcal{C}} := \mathcal{D}_1^2 + \mathcal{D}_2^2 + \mathcal{D}_3^2, \qquad \mathcal{D}_j := \partial_j - \nu_j \partial_\nu, \quad j = 1, 2, 3,$$

is the Laplace-Beltrami operator written in terms of the Günter's tangent derivatives $\mathcal{D}_j$ and $\nu(\omega) = \nu_1(\omega), \nu_2(\omega), \nu_3(\omega)$, $\omega \in \overline{\mathcal{C}}$ is the unit normal vector field on the surface $\mathcal{C}$, while $\partial_\nu = \sum_{j=1}^3 \nu_j \partial_j$ is the normal derivative.

Both BVPs (1) and (2) have unique solutions in the classical setting (in the Sobolev space $\mathbb{H}^1$ provided the data in both BVPs meet the classical constraints: $f, f_0 \in \mathbb{L}_2$ and $q, f^0 + q^0 \in \mathbb{H}^{1/2}$ on corresponding domains.

We prove the following.

THEOREM *Let $q(\omega, \pm h) \in \mathbb{H}^{1/2}(\mathcal{C})$ be uniformly bounded in $\mathbb{L}_2(\mathcal{C})$, $f_h(\omega, t) \to f^0(\omega)$ in $\mathbb{H}^{-1}(\Omega^1)$ and there exists a function $q^0 \in \mathbb{H}^{-1/2}(\mathcal{C})$ such that*

$$\lim_{h\to 0} \frac{1}{2h} \left(\varphi(\cdot), q(\cdot, h) - q(\cdot, -h)\right)_{\mathcal{C}} = \left(\varphi, q^0\right)_{\mathcal{C}}, \qquad \forall\, \varphi \in \mathbb{H}^{1/2}(\mathcal{C}). \tag{3}$$

The scaled energy functional corresponding to the BVP (1) Γ-converges to the energy functional corresponding to the BVP (2).

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