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A BOUNDARY VALUE PROBLEM FOR A CLASS OF HIGHER ORDER NONLINEAR HYPERBOLIC EQUATIONS

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Abstract. A boundary value problem for a class of nonlinear hyperbolic equations of the higher order is considered. The theorems on existence, uniqueness and nonexistence of solutions of this problem are proved.

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On a plane of variables x and t consider the following fourth order hyperbolic equation

$$\square^2 u + f\left(u, \square u\right) = F\left(x, t\right), \tag{1}$$

where $\square := \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$, f and F are given, while u is an unknown scalar function.

Denote by $D_T : 0 < x < t, \quad t < T$ the angular domain bounded by the characteristic segment $\gamma_{1,T} : x = t,\ 0 \le t \le T$, and $\gamma_{2,T} : x = 0, \quad 0 \le t \le T, \gamma_{3,T} : t = T, \quad 0 \le x \le T$, temporal and spatial orientation segments, respectively.

For equation (1) in the domain D_T consider the Darboux type boundary value problem with the following statement: find in the domain D_T a solution $u = u(x,t)$ to equation (1) which satisfies on the parts of the boundary $\gamma_{1,T}$ and $\gamma_{2,T}$ the following conditions

$$u|_{\gamma_{1,T}} = u\left(t,t\right) = 0, \qquad \frac{\partial u}{\partial \nu}|_{\gamma_{1,T}} = \frac{\partial u}{\partial \nu}\left(t,t\right) = 0, \qquad 0 \le t \le T, \tag{2}$$

$$u|_{\gamma_{2,T}} = u\left(0,t\right) = 0, \qquad \frac{\partial^2 u}{\partial x^2}|_{\gamma_{2,T}} = \frac{\partial^2 u}{\partial x^2}\left(0,t\right) = 0, \quad 0 \le t \le T, \tag{3}$$

where $\nu = (\nu_x, \nu_t)$ is a unit vector of the outer normal to the boundary ∂D_T, and $\partial/\partial\nu$ is the derivative in the direction of the normal.

It should be noted that in the case of a plane of two independent variables, in the theory of differential equations and systems of hyperbolic type and its applications, along with mixed problems, the boundary value problems such as Goursat, Darboux, and other problems of this type are no less important. In some ways these problems can be thought of as a limit case for mixed problems when the initial data carrier converges at a single point. These problems describe the process of gas absorption by the sorbent, the harmonic oscillation of a wedge in the supersonic flow, the oscillation of a wire in the viscous liquid, and so on [1] – [3].

For the following kind of second−order hyperbolic equation

$$u_{tt} - u_{xx} + a(x,t)\,u_x + b(x,t)\,u_t + f(x,t,u) = F(x,t)$$

in the linear case (i.e., when the function $f(x,t,u)$ is linear with respect to u), the Goursat and Darboux type problems in the angular domain D_T, when on the part of the boundary $\gamma_{1,T}$ and $\gamma_{2,T}$ are given the Dirichlet or Neuman conditions are well studied (see, for example, [4] − [7] and literature cited there) and their correctness is proved. In the nonlinear case, however, the study of these problems additionally faces substantial difficulties. In some cases, new effects occur, especially when the power nonlinearity in the equation with respect to the sought solution is greater than one [8] − [10]. The novelty that can arise in the nonlinear case consists in the violation of global solvability.

Remark 1. Let $f \in C(R^2)$ and $F \in C\left(\overline{D}_T\right)$. If u, where $u, u\square \in C^2\left(\overline{D}_T\right)$, represents a classical solution of problem (1) − (3), then by introducing the function $v = \square u$ this problem with respect to unknown functions u and v can be reduced to the following boundary value problem

$$L_1(u,v) := \square u - v = 0, \qquad (x,t) \in D_T, \tag{4}$$

$$L_2(u,v) := \square v + f(u,v) = F(x,t), \qquad (x,t) \in D_T, \tag{5}$$

$$u_{\gamma_{1,T}} = u(t,t) = 0, \qquad u_{\gamma_{2,T}} = u(0,t) = 0, \qquad 0 \le t \le T, \tag{6}$$

$$v_{\gamma_{1,T}} = v(t,t) = 0, \qquad v_{\gamma_{2,T}} = v(0,t) = 0, \qquad 0 \le t \le T. \tag{7}$$

On the contrary, if $u, v \in C^2\left(\overline{D}_T\right)$, represents the classical solution of problem (4)-(7), then the function u will be the classical solution of problem (1)-(3).

Definition 1. Let $f \in C(R^2)$ and $F \in C\left(\overline{D}_T\right)$. The system of functions u and v is called a generalized solution of the class C to problem (4) − (7) if $u, v \in C\left(\overline{D}_T\right)$ and there exist sequences

$$u_n, v_n \in \dot{C}^2\left(\overline{D}_T\right) := \left\{ w \in C^2\left(\overline{D}_T\right) : w|_{\gamma_{i,T}} = 0, \quad i = 1,2 \right\} \tag{8}$$

such that

$$\lim_{n\to\infty} \|u_n - u\|_{C(\overline{D}_T)} = 0, \qquad \lim_{n\to\infty} \|v_n - v\|_{C(\overline{D}_T)} = 0, \tag{9}$$

$$\lim_{n\to\infty} \|L_1(u_n,v_n)\|_{C(\overline{D}_T)} = 0, \qquad \lim_{n\to\infty} \|L_2(u_n,v_n) - F\|_{C(\overline{D}_T)} = 0. \tag{10}$$

Remark 2. It is clear that the classical solution $u, v \in C^2\left(\overline{D}_T\right)$ of problem (4) − (7) is a generalized solution of the class C of this problem.

Consider the question of the uniqueness of a solution of problem (4) − (7).

Definition 2. A function $h = h(y)$, $y \in R^n$ is said to satisfy the local Lipschitz condition if $\forall r = const > 0$

$$|h(y_2) - h(y_1)| \le A(r) \|y_2 - y_1\|_{R^n} \quad \forall y_1, y_2 \in R^n \ : \|y_i\|_{R^n} \le r, \quad i = 1,\ 2, \tag{11}$$

where $A = A(r) = const \ge 0$.

Remark 3. It is clear that if $h \in (R^n)$, then condition (11) is satisfied, where, by the Lagrange theorem, in this inequality we can take $A(r) = \max_{\|y_i\|_{R^n} \le r} \|\nabla h\ (y)\|_{R^n}$, $\nabla = \left(\frac{\partial}{\partial y_1},\ \ldots,\ \frac{\partial}{\partial y_n}\right)$.

Theorem 1 (*uniqueness of solution*). *Let $f \in C(R^2)$, $F \in C(\overline{D}_T)$. If the function f satisfies the local Lipschitz condition* (11), *then problem* $(4)-(7)$ *cannot have more than one generalized solution of the class C and thus problem* $(1)-(3)$ *cannot have more than one classical solution.*

Consider the following condition imposed on the function $f(u,v)$:

$$|f(u,v)| \le M_0 + M_1 |u| + M_2 |v| \qquad \forall u, v \in R, \tag{12}$$

where $M_i = const \ge 0$, $i = 0,\ 1,\ 2$.

It is proved that if $f \in C(R^2), F \in C(\overline{D}_T)$ and condition (12) holds, then the following a priori estimates are valid for any generalized solution u, v of the class C of problem $(4)-(7)$

$$|u(x,t)| \le C_1 \|F\|_{L_2(D_t)} + C_2, \quad (x,t) \in D_T, \tag{13}$$

$$|v(x,t)| \le C_3 \|F\|_{L_2(D_t)} + C_4, \quad (x,t) \in D_T, \tag{14}$$

where values $C_i = C_i(t) \ge 0$, $i = 1, \ldots, 4$, do not depend on the functions u, v and F.

Taking into account Remark 3, using a priori estimates (13), (14) and the Leray–Schauder fixed point principle, the following theorem is proved.

Theorem 2 (*existence and uniqueness of the solution*). *Let the function $f \in C^1(R^2)$ satisfy condition* (12). *Then for any function $F \in C^1(\overline{D}_T)$ problem* $(4)-(7)$ *has a unique generalized solution u, v of the class C, which is also a classical solution of this problem and u is a classical solution of problem* $(1)-(3)$ *in the domain D_T.*

Now consider the following special case of equation (1), when

$$f(u,v) = \mu sin u + \lambda e^v, \qquad \lambda, \mu = const. \tag{15}$$

The function $f(u,v)$ given by equation (15) does not satisfy condition (12) for $\lambda \ne 0$, since it contains a strong non-linearity in the form of the term λe^v. Nevertheless, it was proved in [22] that if $\lambda \ge 0$, then for any function $F \in C^1(\overline{D}_T)$ problem $(4)-(7)$ has a unique generalized solution of the class C, which is also a classical solution in the domain D_T.

When $\lambda < 0$, $\mu = const$, the following theorem holds.

Theorem 3 (*absence of solution*). *Suppose the function $f(u,v)$ is given by equality* (15), *where $\lambda < 0$. Then there exists a function $F \in C^1(\overline{D}_T)$ such that problem* $(4)-(7)$ *does not have a generalized solution of the class C and problem* $(1)-(3)$ *does not have a classical solution in the domain D_T.*

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