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ON THE APPROXIMATE SOLUTION OF THE BOUNDARY VALUE PROBLEM FOR ORDINARY DIFFERENTIAL EQUATIONS

Tamaz Vashakmadze Giorgi Buzhghulashvili

Abstract. The problem of approximate solution for ordinary second-order nonlinear differential equations with Sturm- Liouville boundary conditions by the multi-point difference method, when the sufficient conditions for the existence and uniqueness of the solution are fulfilled according to [1], will be studied. The method developed in [1] and its modification is used to find both -the solution and its derivative- for a class of less smooth functions. The case when the right-hand side of the differential equation is an oscillating type function is studied separately. In this case, the differential analog solution is performed at each iteration stage by summing the finite part of the trigonometric series with variable coefficients, which is related to the selection of the optimal method during the numerical implementation of the "Fast Fourier Transform".

Keywords and phrases: Multi-point difference method, class of less smooth functions, trigonometric series with variable coefficients.

AMS subject classification (2010): 65M60, 65M99.

1. This work presents the results of research on the construction and study of difference schemes for traditional problems of mathematical physics, the accuracy of which is consistent with the smoothness of the generalized solution of the initial differential problem and represents an expansion of our reports on XXXVII International Enlarged Sessions of the Seminar of VIAM and XI International Conference of the Georgian Mathematical Union.

For the nonlinear differential equation with the monotone operator [1]

$$(k(t)\,u'(t))' = f(x, u(t), u'(t)), \quad u(0) - k_1 u'(0) = \alpha, \quad u(1) + k_2 u'(1) = \beta,$$

the same type of analysis is valid with addition of solving two Cauchy problem from center to the right and left. However, for simplicity and lack of space we provide formulas for the following case

$$y''(x) = f(x, y(t)), \quad u(0) = \alpha, \quad u(1) = \beta.$$

According to [1], let us introduce net. $x_i = (i-1)\,h,\ i = 1, 2, \ldots, 2k+1,\ p = 2s+1,\ h = 1/2ks$

$$y_{(t-1)s+i} = \frac{2s-(i-1)}{2s} y_{(t-1)s+1} + \frac{i-1}{2s} y_{(t+1)s+1} + I_{(t-1)s+i},$$
$$(t = 1, 2, \ldots, 2k-1, i = 2, 3, \ldots, s+1), \ (t = 2k-1, i = 2, 3, \ldots, 2s).$$

For simplicity consider s=1. We have the following scheme, with rule of computing y_{k+1} and recurrence relations for other nodes

$$y_i = \frac{1}{2}y_{i-1} + \frac{1}{2}y_{i+1} + I_i \ \ , \ i = 2, \dots \ , \ 2k, \quad y_{k+1} = \frac{1}{2}y\left(0\right) + \frac{1}{2}y\left(1\right) + \sigma_{k+1},$$

$$\sigma_{k+1} = \sum_{t=1}^{k-1} t \ I_{t+1} + k \ I_{k+1} + \sum_{t=1}^{k-1} tI_{2k-t},$$

$$y_{t+1} = \frac{t}{t+1}y_{(t+1)+1} + \frac{1}{(t+1)}y\left(0\right) + \Sigma^{[t]}, \ \Sigma^{[t]} = \frac{2}{t+1}\sum_{r=1}^{t} r \ I_{1+r},$$

$$y_{(t+1)s+1} = \frac{t}{t+1}y_{ts+1} + \frac{1}{(t+1)}y\left(1\right) + \Sigma^{[2k-t]}, \quad \Sigma^{[2k-t]} = \frac{2}{t+1}\sum_{r=1}^{t} rI_{2k-t},$$

$$y_{t+1} = \frac{2k-t}{2k}y\left(0\right) + \frac{t}{2k}y\left(1\right) + \sigma_{t+1}, \quad \sigma_{t+1} \leq \left[1 - \left(\frac{k-t}{k}\right)^2\right].$$

We note that $I_i = \int_{x_{i-1}}^{x_{i+1}} G\left(x_i, t\right) y''\left(t\right) dt = \frac{h^2}{2} f\left(x_i, y\left(x_i\right)\right) + 2r$, where r is a remainder term and G is Green's function [1]. We consider that $f\left(x, y\left(t\right)\right)$ has a finite number of jump discontinuities with respect to first variable, and satisfies Lipshitz conditions with respect to the second variable. In this case provided scheme is of compression type, thus, depending on the number of jump discontinuities, for a sufficiently large value of k, by the fixed point iteration with the above algorithm $y \in W_2^1 = \left\{v(x) : v(x) \in L_2, v'(x) \in L_2, x \in \Omega\right\}, \Omega = (0, 1)$ is found.

For illustrating the above given material consider the following numerical example :

$$y''\left(x\right) = \frac{2y}{\left(x+1\right)^2} \ + 200 \ \left(x^2 - 1\right) H\left(x\right), \quad y\left(0\right) = \frac{1}{3}, y\left(1\right) = \frac{1}{4},$$

$$H\left(x\right) = \sum_{i=1}^{25} S\left(x - \frac{2i-2}{50}; \frac{1}{50}\right) - S\left(x - \frac{2i-1}{50}; \frac{1}{50}\right),$$

$$\theta(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}, \qquad S\left(x; h\right) = \theta\left(x\right) - \theta\left(x - h\right).$$

$\theta(x)$ is a unit step function at $x = 0$, and $S\left(x; h\right)$ is 1 when $0 < x \leq h$ and 0 otherwise. On the left figure is depicted graph of function $200 \ \left(x^2 - 1\right) H\left(x\right)$, and on the right figure its graph of an analytic solution. We provide only graph because formula for analytic solution would take too much space. And on the table below we provide maximum value of absolute error of the approximation to the real solution on m=20-th iteration for different values of k and s.

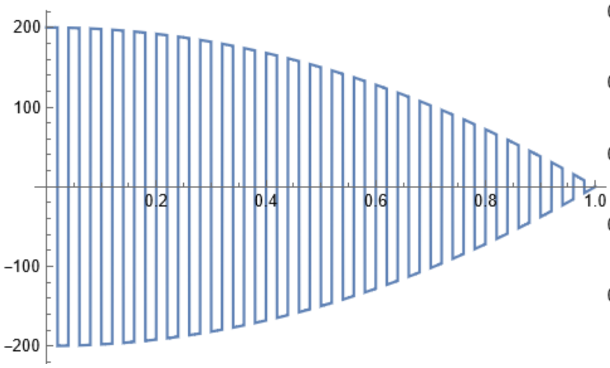

Figure 1: $200(x^2-1)H(x)$

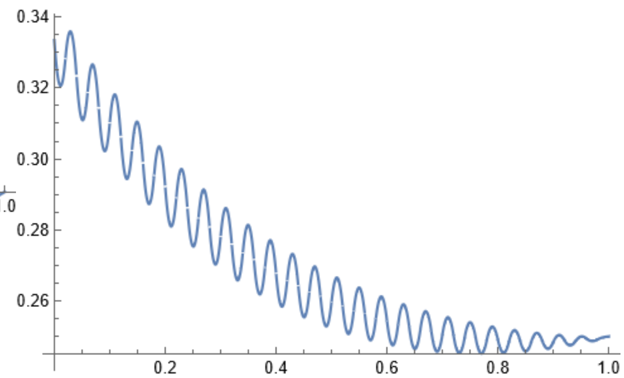

Figure 2: $y(x)$

k\s	1	2	3	4
50	0.0197136	0.00657232	0.00936506	0.00266741
100	0.0098577	0.00328615	0.00468265	0.000943318
150	0.00657197	0.00219077	0.0031218	0.000542124
200	0.00492904	0.00164307	0.00234136	0.000374059
250	0.00394327	0.00131446	0.00187309	0.000283631
300	0.00328607	0.00109538	0.00156091	0.000227683
400	0.00246457	0.000821537	0.00117068	0.000162629
500	0.00197166	0.00065723	0.000936548	0.000126199
700	0.00140834	0.00046945	0.000668964	8.69547e-5
1000	0.000985838	0.000328615	0.000468275	5.91951e-5

2. Let us consider the case when the right part side of the differential equation depends on oscillating functions. For demonstration the difficulties arising in this case it is sufficient to study some characteristic properties of the following boundary value problem.

$$y''(x) = \sin\left(\pi\omega\ (y(x) - x\ )\right) \quad y(0) = 0,\ y(1) = 1. \tag{1}$$

The evident solution is $y(x) = x$ if $\omega \leq 3$. If we applied the methodology of preliminary part, it is necessary to calculate Fourier (trigonometric) type finite sum on each step m of iteration. For finding approximate solution for this problem for each step is same to calculate some integrals of type

$$I_i = \int_{x_{i-1}}^{x_{i+1}} G(x,t)\ \sin\Big(\omega\ (y^{[m]}(x) - x\ )\Big)dt.$$

By $y^{[m]}(x_i)$ we construct cubic spline [2], [3] $y^{[m]}(x_i)$, so that one can compute the Integral I_i using adaptive Clenshaw-Curtis-Kronrod [3]-[5] quadrature formula. Interpolating spline is introduced so that for arbitrary ω one might need to reevaluate integral if desired precision is not yet reached with higher order of the rule. For values of ω near 3 and above on initial iterations the approximation is converging towards solution but after certain accuracy is reached approximation diverges from the solution, we suppose that bifurcation takes place and needs further research.Below we provide table of numerical results for (1) on different values of $\pi\omega$. Note that for case 4π on 10-th iteration diverging process described above takes place.

m\ $\pi\omega$	π	2π	3π	4π
1	0.0253303	0.0112579	0.00633257	0.00405285
2	0.00401822	0.00119276	0.000503516	0.000257876
3	0.000639533	0.000126559	4.00694e-05	1.64175e-05
4	0.000101786	1.34285e-05	3.18863e-06	1.0458e-06
5	1.61998e-05	1.42491e-06	2.53745e-07	6.75777e-08
6	2.57828e-06	1.5128e-07	2.01931e-08	5.89354e-09
7	4.10345e-07	1.61391e-08	1.60714e-09	3.11607e-09
8	6.53085e-08	1.79627e-09	1.27964e-10	5.17308e-09
9	1.03942e-08	2.70669e-10	1.02061e-11	8.25944e-09
10	1.65428e-09	1.05188e-10	8.19498e-13	1.3147e-08

BVP (1) is sufficient for demonstrating that within conditions of $\omega \leq 3$, the same process is valid for BVPs of type $y''(x) = \sin^{\gamma}(\pi\omega\ (y(x) - x\ )) + \varepsilon\, y^2(x)\,, u(0) - k_1 u'(0) = \alpha, \quad u(1) + k_2 u'(1) = \beta$ where ε is small, $\gamma \leq 2$.

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Author(s) address(es):

Tamaz Vashakmadze
I.Vekua Institute of Applied Mathematics, I. Javakhishvili Tbilisi State University
University str. 2, 0186 Tbilisi, Georgia
E-mail: tamazvashakmadze@gmail.com

Giorgi Buzhghulashvili
Department of Mathematics, I. Javakhishvili Tbilisi State University
University str. 2, 0186 Tbilisi, Georgia
E-mail: giorgi.buzhghulashvili570@ens.tsu.edu.ge