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NUMERICAL SOLUTION FOR J. BALL'S BEAM EQUATION WITH VELOCITY-DEPENDENT EFFECTIVE VISCOSITY

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Abstract. A mathematical model is formulated for an initial-boundary value problem associated with the J. Ball integro-differential equation, which serves as a mathematical description of the dynamic state exhibited by a beam. The solution to this problem is approximated through a combination of the Galerkin method, a stable symmetrical difference scheme, and the Jacobi iteration method. This paper desires to present an approximate solution to a practical problem, specifically focusing on the numerical results obtained from the initial-boundary value problem pertaining to a specific iron beam. Notably, the effective viscosity of the material is considered to be dependent on its velocity.

Keywords and phrases: Nonlinear dynamic beam equation, J. Ball equation, Galerkin method, Implicit symmetric difference scheme, Jacobi iterative method, Iron beam, Numerical realization.

AMS subject classification (2010): 65M60, 65M06, 65Q10, 65M15.

1 Statement of the problem. Let us consider the nonlinear equation

$$
\begin{gathered}
u_{tt}(x,t)+\delta u_t(x,t)+\gamma u_{xxxxt}(x,t)+\alpha u_{xxxx}(x,t) \\
-\left(\beta+\kappa\int_0^L u_x^2(x,t)\,dx\right)u_{xx}(x,t)-\sigma\left(\int_0^L u_x(x,t)\,u_{xt}(x,t)\,dx\right) \\
\times u_{xx}(x,t)=f(x,t), \qquad 0<x<L, \qquad 0<t\le T,
\end{gathered} \tag{1}
$$

with the initial boundary conditions

$$
u(x,0)=u^0(x), \quad u_t(x,0)=u^1(x), \tag{2}
$$

$$
u(0,t)=u(L,t)=0, \quad u_{xx}(0,t)=u_{xx}(L,t)=0. \tag{3}
$$

In the given context, let $\alpha,\gamma,\kappa,\sigma,\beta$, and let δ be constants, where the first four are positive numbers. Furthermore, consider the functions $u^0(x)\in W_2^2(0,L)$ and $u^1(x)\in L_2(0,L)$, satisfying the conditions $u^0(0)=u^1(0)=u^0(L)=u^1(L)=0$. The right-hand side function $f(x,t)$ belongs to $L_2((0,L)\times(0,T))$. We assume the existence of a solution $u(x,t)\in W_2^2((0,L)\times(0,T))$ for the problem (1)-(3).

The present article serves as a direct continuation of previous works [1]-[4], which focused on developing algorithms and performing corresponding numerical computations for approximating solutions to nonlinear integro-differential equations of the Timoshenko type. In this particular study, we address an initial-boundary value problem associated with the J. Ball integro-differential equation, which characterizes the dynamic state of a beam (see [5]). To approximate the solution, we employ the Galerkin method, a stable symmetric difference scheme, and the Jacobi iteration method. The algorithms proposed in [2]-[3] have been validated through various tests. Additionally, this article, along with [4], presents an approximate solution to a practical problem. Specifically, we provide numerical results for the initial-boundary value problem concerning an iron beam, which are presented in a tabular form.

The physical model utilized by J. Ball in his publication [5] is derived from the Handbook of Engineering Mechanics, authored by E. Mettler (see [6]). In this model, the corresponding initial-boundary value problem for the integro-differential equation governing the behaviour of a beam (denoted as equation (1)) is formulated. The constants α, γ, κ, σ, β, and δ present in the problem are defined as follows:

$$\alpha = \frac{E \cdot I}{\rho}, \quad \beta = \frac{E \cdot A \cdot \Delta}{L \cdot \rho}, \quad \gamma = \frac{\eta \cdot I}{\rho}, \quad \kappa = \frac{E \cdot A}{2L \cdot \rho}, \quad \sigma = \frac{A\eta}{L \cdot \rho}.$$

Here, E denotes Young's modulus, A represents the cross-sectional area, η signifies the effective viscosity, I stands for the cross-sectional second moment of area, ρ corresponds to the mass per unit length in the reference configuration, L symbolizes the length of the beam, Δ signifies the extension or change in the beam length, and δ refers to the coefficient of external damping.

2 The numerical realization. To approximate the solutions to initial-boundary value problems (1)-(3), a collection of programs was developed within the Maple software environment. Subsequently, several numerical experiments were conducted to facilitate this approximation process. The purpose of this paper is to present an approximate solution to a practical problem. Specifically, the tables in this paper illustrate the results obtained from numerical computations of the initial-boundary value problem concerning an iron beam.

The present study investigates the issues pertaining to the initial-boundary value problem of an iron beam, considering the following parameter values: length $L = 1\,\mathrm{m}$, time $T = 1\,\mathrm{sec}$, temporal grid length $\tau = 0.05\,\mathrm{sec}$, number of coordinate functions in the Galerkin method $n = 5$, number of iterations $n_{iter} = 5$, Young's modulus $E = 1.9 \times 10^6\,{}^{\mathrm{kg}}\!/_{\mathrm{ms}^2}$, density $\rho = 7.874\,{}^{\mathrm{g}}\!/_{\mathrm{cm}^3}$, spatial step size $\Delta = 0.01\,\mathrm{m}$, cross-sectional area $A = 0.01\,\mathrm{m}^2$, moment of inertia $I = 1000\,\mathrm{Pa}$.

In our investigation, we consider the case where the velocity-dependent effective viscosity has the form $\eta = (V_0 + K_v \cdot t)^{-1}$, with initial velocity $V_0 = 5\,{}^{\mathrm{m}}\!/_{\mathrm{sec}}$ and coefficient $K_v = 2$. The time variable t is bounded within the interval $[0, 1]$. Additional parameters are defined as follows: $\alpha = 0.24613 \times 10^6 \cdot I$, $\beta = 241.3$, $\gamma = 0.12954 \times I \cdot \eta$, $\kappa = 12065$, $\sigma = 0.0127 \times \eta$, and $\delta = 0$. The initial functions are specified as $u^0(x) = \sin\left(\frac{\pi x}{L}\right)$ and $u^1(x) = 0$, while the right-hand side function is denoted as $f(x,t) \equiv 0$.

As the velocity increases, the viscosity naturally decreases, resulting in an increase in the bends, denoted as $u(x,t)$, as confirmed by the numerical experiments presented in

Table 1. In our study, we considered two main cases: (a) Simple model: In this approach, we calculate the value of η for each specific t and obtain constant coefficients for all time layers in the corresponding difference equations. (b) Complex model: In this case, the coefficients in the difference equations depend on t for all time layers. For the simple model, we specifically examined three cases: Case 1: $t = 0$, $\eta = 1/5$. Case 2: $t = 0.5$, $\eta = 1/6$. Case 3: $t = 1$, $\eta = 1/7$. We compared the results obtained from these three cases with those derived from the complex model (Case 4), considering different values for the spatial and temporal variables.

$t\backslash x$	Case	$x = 0$	$x = 0.2$	$x = 0.4$	$x = 0.6$	$x = 0.8$	$x = 1$
$t = 0$	1	0	0.587785	0.951057	0.951057	0.587785	0
$t = 0$	2	0	0.587785	0.951057	0.951057	0.587785	0
$t = 0$	3	0	0.587785	0.951057	0.951057	0.587785	0
$t = 0$	4	0	0.587785	0.951057	0.951057	0.587785	0
$t = 0.25$	1	0	5.289964	8.559341	8.559341	5.289964	0
$t = 0.25$	2	0	5.289980	8.559368	8.559368	5.289980	0
$t = 0.25$	3	0	5.289992	8.559387	8.559387	5.289992	0
$t = 0.25$	4	0	5.289967	8.559347	8.559347	5.289967	0
$t = 0.5$	1	0	11.167430	18.069281	18.069281	11.167430	0
$t = 0.5$	2	0	11.167504	18.069401	18.069401	11.167504	0
$t = 0.5$	3	0	11.167557	18.069487	18.069487	11.167557	0
$t = 0.5$	4	0	11.167459	18.069329	18.069329	11.167459	0
$t = 0.75$	1	0	17.044574	27.578700	27.578700	17.044574	0
$t = 0.75$	2	0	17.044747	27.578980	27.578980	17.044747	0
$t = 0.75$	3	0	17.044871	27.579180	27.579180	17.044871	0
$t = 0.75$	4	0	17.044670	27.578856	27.578856	17.044670	0
$t = 1$	1	0	22.921357	37.087535	37.087535	22.921357	0
$t = 1$	2	0	22.921670	37.088041	37.088041	22.921670	0
$t = 1$	3	0	22.921894	37.088403	37.088403	22.921894	0
$t = 1$	4	0	22.921577	37.087891	37.087891	22.921577	0

Table 1.

Conclusion

Based on the observed numerical experiments, it is evident that as the effective viscosity, denoted by η, increases (or decreases), the corresponding numerical values of the displacement function, $u(x, t)$, for specific values of x and t exhibit a decreasing (or increasing) trend. Specifically, when considering the case of velocity-dependent effective viscosity, an increase in velocity leads to a decrease in viscosity, resulting in amplified deflections (or bending) of the beam. Furthermore, for a fixed value of η, the numerical values of the displacement function for a given x tend to increase as time t progresses. Notably, the numerical values of the displacement function at a particular t exhibit symmetry with respect to the midpoint of the beam, located at $x = L/2$.

REFERENCES

1. Papukashvili, G., Peradze, J., Tsiklauri, Z. On a stage of a numerical algorithm for Timoshenko type nonlinear equation. *Proc. A. Razmadze Math. Inst., Tbilisi* **158** (2012), 67-77.
2. Papukashvili, A., Papukashvili, G., Sharikadze, M. Numerical calculations of the J. Ball nonlinear dynamic beam. *Rep. Enlarged Sess. Semin. I. Vekua Inst. Appl. Math.*, **32** (2018), 47-50.
3. Papukashvili, A., Papukashvili, G., Sharikadze, M. On a numerical realization for a Timoshenko type nonlinear beam equation. *Rep. Enlarged Sess. Semin. I. Vekua Inst. Appl. Math.*, **33** (2019), 51-54.
4. Papukashvili, A., Geladze, G., Vashakidze, Z., Sharikadze, M. On the Algorithm of an Approximate Solution and Numerical Computations for J. Ball Nonlinear Integro-Differential Equation. *Rep. Enlarged Sess. Semin. I. Vekua Inst. Appl. Math.*, **36** (2022), 75-78.
5. Ball, J.M. Stability theory for an extensible beam. *J. Differential Equations*, **41** (1973), 399-418.
6. Mettler, E. Dynamic buckling, in Handbook of Engineering Mechanics (S. Flugge, Ed.). *Chapter 62, Mc Graw-Hill, New York, 1962.*

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