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ABSOLUTE CONVERGENCE OF DOUBLE FOURIER TRIGONOMETRIC SERIES

Rusudan Meskhia

Abstract. The sufficient conditions for the generalized absolute convergence of double Fourier trigonometric series are established in terms of mixed and partial moduli of δ-variation of the function of two variables.

Keywords and phrases: Double Fourier trigonometric series, generalized absolute convergence, the modulus of δ-variation.

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1 Introduction. The classical results of Bernstein, Szasz, Zygmund, related to the absolute convergence of single trigonometric Fourier series are well known [1]. The questions dealing with the absolute convergence of Fourier trigonometric series have been investigated in the works Z. Chanturia [2], T. Karchava [5], F. Moricz and A. Veres [7] and many other authors.

2 Content. The problem of convergence of the series

$$\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\gamma_{mn}|\widehat{f}_{m,n}|^r,\ \ 0<r<2,$$

is considered, where $\{\gamma_{mn}\}_{m\geq 1,\,n\geq 1}$ is a defined multiple sequence of nonnegative numbers and

$$\widehat{f}(m,n)=\frac{1}{4\pi^2}\iint\limits_{T^2} f(x,y)e^{-i(mx+ny)}\,dx\,dy,\ \ (m,n)\in\mathbb{Z}^2,$$

are the Fourier trigonometric coefficients of the function $f(x,y)\in L_1(T^2)$, where $T^2=T\times T$, $T=[-\pi,\pi]$, $f(x,y)$ is a complex-valued function periodic with period 2π in each variable and the double Fourier series of f is given by

$$f(x,y)\sim\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\widehat{f}(m,n)e^{i(mx+ny)},\ \ (x,y)\in T^2.$$

Following the definition in [3] and notations of [7] a sequence $\{\gamma_{kj}\}_{k\geq 1,\,j\geq 1}$, $k,j\in\mathbb{N}$, of nonnegative numbers is said to belong to the class A_α, for some $\alpha\geq 1$, if

$$\Big(\sum_{k\in D_m}\sum_{j\in D_n}\gamma_{kj}^{\alpha}\Big)^{\frac{1}{\alpha}}\leq c\cdot 2^{\frac{(m+n)(1-\alpha)}{\alpha}}\sum_{k\in D_{m-1}}\sum_{j\in D_{n-1}}\gamma_{kj},$$

$$\Big(\sum_{k\in D_m}\gamma_{k1}^{\alpha}\Big)^{\frac{1}{\alpha}}\leq c_1\cdot 2^{\frac{m(1-\alpha)}{\alpha}}\sum_{k\in D_{m-1}}\gamma_{k1},$$

$$\Big(\sum_{j\in D_n}\gamma_{1j}^{\alpha}\Big)^{\frac{1}{\alpha}}\leq c_2\cdot 2^{\frac{n(1-\alpha)}{\alpha}}\sum_{j\in D_{n-1}}\gamma_{1j},$$

where $D_0 = D_{-1} = \{1\}$, $D_i = \{2^{i-1}+1, 2^{i-1}+2, \dots, 2^i\}$, $i \in \mathbb{N}$, and the constants c, c_1, c_2 depend only on α. We agree to put

$$\gamma_{-k,j} = \gamma_{k,-j} = \gamma_{-k,-j} = \gamma_{kj}, \quad k, j \in \mathbb{N}.$$

$B(T^2)$ denotes a class of bounded functions on T^2.

$BV_s(T^2)$, $s \geq 1$, is the class of the functions with the bounded s variation on T^2 [7].

$C(T^2)$ is the class of continuous functions on T^2.

$\varphi(m, n; \delta_1, \delta_2; f)$ denotes the mixed modulus of $\delta(\delta_1; \delta_2)$-variation of the function $f \in B(T^2)$, $\varphi_1(m; \delta_1; f)$ and $\varphi_2(m; \delta_2; f)$ are partial moduli of δ-variation. The mixed and partial moduli of the function $f(x, y) \in B(T^2)$ are defined, according to Karchava's [5] modulus of δ-variation, in the following way:

$$\varphi(m, n; \delta_1, \delta_2; f) = \sup_{\Pi_{m,n;\delta_1,\delta_2}} \sum_{k=1}^{m} \sum_{j=1}^{n} \omega(f; I_k \times B_j),$$

$$\varphi_1(m; \delta_1; f) = \sup_{-\pi \leq y \leq \pi} \sup_{\Pi_{m;\delta_1}} \sum_{k=1}^{m} \omega_1(f; I_k),$$

$$\varphi_2(n; \delta_2; f) = \sup_{-\pi \leq x \leq \pi} \sup_{\Pi_{n;\delta_2}} \sum_{j=1}^{n} \omega_2(f; B_j),$$

where $m, n \in \mathbb{N}$, $\delta_1, \delta_2 > 0$,

$$\omega(f; I_k \times B_j) = \sup \Big\{ \big| f(x+h_1, y+h_2) - f(x, y+h_2) - f(x+h_1, y) + f(x, y) \big| : \\ (x, y), (x+h_1, y+h_2) \in I_k \times B_j, \quad h_1, h_2 > 0 \Big\},$$

$$\omega_1(f; I_k) = \sup \Big\{ \big| f(x+h_1, y) - f(x, y \big| : \quad x, x+h_1 \in I_k, \quad h_1 > 0 \Big\},$$

$$\omega_2(f; B_j) = \sup \Big\{ \big| f(x, y+h_2) - f(x, y) \big| : \quad y, y+h_2 \in B_j, \quad h_2 > 0 \Big\},$$

$\Pi_{m,n;\delta,\delta_2}$ is an arbitrary system of mn pairwise nonintersecting rectangles $I_k \times B_j \subset T^2$, $1 \leq k \leq m$, $1 \leq j \leq n$, $k, j \in \mathbb{N}$.

$\Pi_{m;\delta_1}$ ($\Pi_{n;\delta_2}$) is an arbitrary system of nonintersecting intervals $\{I_k\}_{1 \leq k \leq m}$ ($\{B_j\}_{1 \leq j \leq n}$) of the segment $[-\pi; \pi]$. The length of each interval I_k (B_j) is equal to δ_1 (δ_2).

The following statement is true

Theorem. *Let* $\{\gamma_{kj}\} \in A_{\frac{2}{2-r}}$, $0 < r < 2$, $f(x, y) \in B(T^2)$ *be the function satisfying the* *following conditions:*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\gamma_{mn}(mn)^{-r}\Big(\sum_{k=1}^{m}\sum_{j=1}^{n}\frac{\varphi^2(k,j;\frac{1}{m},\frac{1}{n};f)}{k^2j^2}\Big)^{\frac{r}{2}}<+\infty,$$

$$\sum_{m=1}^{\infty}\gamma_{m1}m^{-r}\Big(\sum_{k=1}^{m}\frac{\varphi_1^2(k;\frac{1}{m};f)}{k^2}\Big)^{\frac{r}{2}}<+\infty,$$

$$\sum_{n=1}^{\infty}\gamma_{1n}n^{-r}\Big(\sum_{j=1}^{n}\frac{\varphi_2^2(j;\frac{1}{n};f)}{j^2}\Big)^{\frac{r}{2}}<+\infty,$$

then the series

$$\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\gamma_{mn}|\widehat{f}(m,n)|^r,\quad 0<r<2, \tag{1}$$

is convergent.

This theorem presents the analogue of the theorem, obtained by Meskhia [6] for double Fourier trigonometric series and it was shown that the sufficient condition is unimprovable in a certain sense.

From the theorem follows the next

Corollary. Let $f(x,y)\in C(T^2)\cap BV_s(T^2)$ for some $s\in[1,2)$. If $\gamma=\{\gamma_{mn}\}\in A_{\frac{2}{r-2}}$, $0<r<2$, and

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{-r}\omega^{(2-s)\frac{r}{2}}\Big(f;\frac{1}{m};\frac{1}{n}\Big)<+\infty,$$

then the series (1) is convergent. $\omega(f;\delta_1,\delta_2)$ denotes the modulus of continuity of a function $f(x,y)\in C(T^2)$ and is defined by

$$\omega(f;\delta_1,\delta_2)=\sup\Big\{\big|f(x+h_1,y+h_2)-f(x,y+h_2)-f(x+h_1,y)+f(x,y)\big|:\\ (x,y)\in T^2,\ \ 0<h\le\delta,\ \ 0<h_2\le\delta_2\Big\},\quad \delta_1,\delta_2>0.$$

The corollary was obtained by F. Moricz and A. Veres [7] and represents the analogue of L. Gogoladze and R. Meskhia [4] theorem for the double Fourier trigonometric series.

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Author(s) address(es):

Rusudan Meskhia
Faculty of Exact and Natural Sciences, I. Javakhishvili Tbilisi State University
University str. 13, 0186 Tbilisi, Georgia
E-mail: rusudan.meskhia@tsu.ge