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ON THE EXACT SOLUTIONS OF THE ZAKHAROV-KUZNETSOV DYNAMICAL EQUATION IN AN ELECTRON-POSITRON-ION PLASMA

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Abstract. Using the exp-function method the traveling wave special exact solution of the (2+1)D nonlinear Zakharov-Kuznetsov partial differential equation in an electron-positron-ion plasma is represented. The result is expressed in the form of the hyperbolic function and has a spatially isolated structural form. Traveling wave velocity is defined as the function of dynamic parameters.

Keywords and phrases: Traveling waves, exp-function method, exact solution, soliton.

AMS subject classification (2010): 76B25, 76D17, 35Q51, 74J40, 76L05.

1 Introduction. Zakharov-Kusnetsov (ZK) equation governs the dynamics of weakly nonlinear ion-acoustic waves in magnetized plasmas. The subject of our investigation is the fully ionized electron-positron-ion (EPI) plasma. The EPI plasmas are found not only in early universe but also in astrophysical environments such as in magnetosphere of pulsars, active galactic nuclei, interstellar medium, etc. EPI plasma can be artificially created in laboratories. The nonlinearity could produce various structures, such as solitons, shocks, peakons, cuspons, etc. The linear and nonlinear properties of the ion-acoustic waves (IAWs) propagating obliquely to an external magnetic field in a weakly relativistic, rotating, and magnetized EPI plasma were investigated by Mushtaq and Shah [1]. The propagation of linear and nonlinear electrostatic waves in a magnetized anisotropic EPI plasma with superthermal electrons and positrons investigated by Adnan et al. [2]. Recently finding of exact solutions of nonlinear evolution partial differential equations attracts more interest of researchers. Among the developed various powerful different methods exp-function method [3] seems more effective and capable for finding exact solutions of ZK type equations. Using the special exp-function method traveling wave exact solutions of 2D nonlinear Burgers equation are obtained by Tsamalashvili [4], showing the existence of spatially isolated structural (soliton-like) forms. Employing the special exp-function expansion method Kaladze et al. [5] constructed exact traveling wave solutions for the ZK modified equal-width (2+1)D equation and presented new special exact solutions justified the existence of solitary structures (solitons, kink-solitons , etc.). To the best of our knowledge, earlier studies did not consider the exact solutions of ion-acoustic structures (solitons and shocks) in three-component EPI plasma. Therefore, the goal of the present work is the extension of the nonlinear ion-acoustic structures (solitons and shocks) in a fully ionized three-component EPI plasma by the presentation of new exact special solutions of the (2+1) dimensional Zakharov-Kuznetsov equation.

2 Solution of (2+1)-dimensional Zakharov-Kuznetsov equation. Consider obtained in [1, 2] nonlinear ZK equation

$$\Phi_t + A(\Phi^2)_x + B\Phi_{xxx} + C\Phi_{xyy} = 0, \tag{1}$$

where A,B, and C are real valued coefficients and Φ describes the electrostatic potential. To find the traveling wave solution we use the following transformation $\Phi = \Phi(\eta)$, $\eta = x + y - Vt$, then we can convert Eq.(1) into the following ODE

$$-V\Phi' + (B + C)\Phi''' + 2A\Phi\Phi' = 0, \tag{2}$$

where the prime denotes the derivatives with respect to η. Now integrating Eq. (2), we have

$$-V\Phi + (B + C)\Phi'' + A\Phi^2 + c_1 = 0. \tag{3}$$

Here c_1 is the integration constant. The use of special exp-function expansion method [4,5] allows the possibility to find exact solutions of a nonlinear evolutionary equation (3) by the series of function $\exp(-n\varphi(\eta))$, where the function $\varphi(\eta)$ satisfies the ODE:

$$\frac{d\varphi(\eta)}{d\eta} = e^{-\varphi(\eta)} + \mu e^{\varphi(\eta)} + \lambda, \tag{4}$$

where, μ and λ are parameters.Thus, we are seeking the solution of Eq.(3) by the following finite series

$$\Phi(\eta) = \sum_{n=0}^{M} a_n \exp(-n(\varphi(\eta)), \tag{5}$$

where a_n and $0 \leq n \leq M$ are constants and M is homogenous balance number. It is clear that the solution of Eq.(4) depends on the relations between μ and λ. Namely, we will consider the following special case when when $\lambda^2 - 4\mu > 0, \mu \neq 0$, when the solution of Eq.(4) can be given as

$$\varphi(\eta) = \ln\left\{\frac{1}{2\mu}\left[-\sqrt{\lambda^2 - 4\mu}\tanh(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}(\eta + c_2)) - \lambda\right]\right\}, \tag{6}$$

where c_2 is an integration constant. The positive integer M can be determined by considering the homogeneous balance between the highest order derivatives Φ'' and nonlinear terms Φ^2 appearing in Eq.(3).Then we get $M = 2$. Thus, the trial solution of Eq.(3) can be stated as

$$\Phi(\eta) = a_0 + a_1 e^{-\varphi(\eta)} + a_2 e^{-2\varphi(\eta)}, \tag{7}$$

where $a_2 \neq 0$, a_1 and a_2 are constants. Substituting Φ, along with (4) into ODE (3) and equating the coefficients of all powers of $\exp(-n\varphi)$ to zero, yields a set of algebraic equations for unknown coefficients a_n and parameters μ, λ, V

$$\begin{cases} -Va_0 + (B+C)(a_1\lambda\mu + 2a_2\mu^2) + a_0^2 A + c_1 = 0, \\ -Va_1 + (B+C)(2a_1\mu + 6a_2\lambda\mu + \lambda^2 a_1) + 2Aa_0a_1 = 0, \\ -Va_2 + (B+C)(3\lambda a_1 + 4a_2\lambda^2 + 8\mu a_2) + A(a_1^2 + 2a_0a_2) = 0, \\ (B+C)(2a_1 + 10a_2\lambda) + 2Aa_1a_2 = 0, \\ 6(B+C)a_2 + Aa_2^2 = 0. \end{cases} \tag{8}$$

By solving the algebraic system (8), we define

$$a_0 = \frac{1}{2A}[V - (B+C)(\lambda^2 + 8\mu)], \quad a_1 = -6\frac{\lambda}{A}(B+C) = \lambda a_2, \quad a_2 = -\frac{6}{A}(B+C). \tag{9}$$

Note that the second equation of the system (8) is automatically satisfied. Further, we assume that $B + C \neq 0$, and from the first equation of (8), we get

$$\lambda^2 - 4\mu = \frac{\sqrt{V^2 - 4Ac_1}}{|B+C|} \tag{10}$$

and the solution (7) becomes

The graph of this solution with chosen parameters is given in Fig.1showing a row of solitons. Number of solitons is regulated by the value of V.

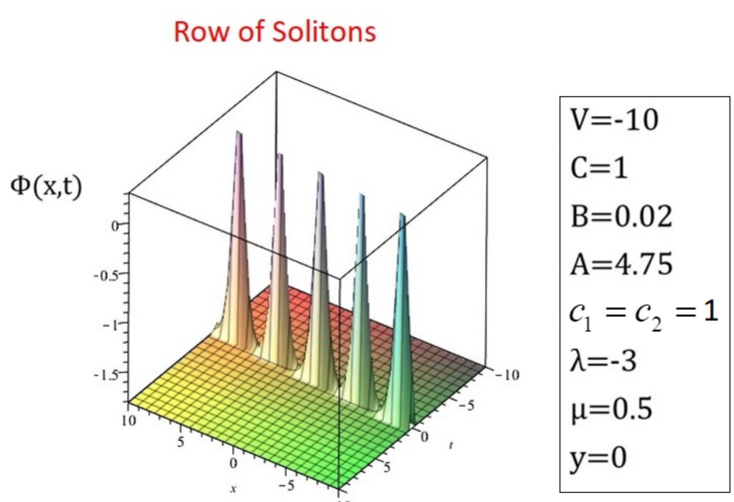

Figure 1

3 Summary. The $\exp(-\varphi(\eta))$-expansion method has been successfully applied to find the exact traveling wave solution of the (2+1)D nonlinear ZK equation (1). Obtained solution (11) is quite similar (but not the same) to the solution obtained in [5] for ZK modified equal-width (2+1)D equation: as it was expected it became less sensitive with the traveling velocity V and represents solitary wave solution of soliton forms expressed through the hyperbolic tangent function. Solution (11) has been verified by substituting back into the original Eq. (1) and found correct.

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