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NONLINEAR FILTERING PROBLEM AND MARTINGALE REPRESENTATION *

Vakhtang Jaoshvili   Valeriane Jokhadze   Omar Purtukhia

Abstract. We study functionals whose filter is not stochastically smooth and propose a method for finding the integrand. The question of representing Brownian functionals as a stochastic Itô integral with an explicit form of the integrand is investigated. The class of functionals under consideration also includes functionals that are not smooth in the sense of Malliavin, to which both the well-known Clark-Ocone formula (1984) and its generalization, the Glonti-Purtukhia representation (2017), are inapplicable.

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1 Introduction. If ξ_t is the solution of a stochastic differential equation and f is a C^2-function, then according to Ito's formula $f(\xi_t)$ is a semimartingale and hence, under appropriate conditions, $E[f(\xi_t)|\Im_t^\eta]$ is a right-continuous semimartingale with respect to σ-algebras $\Im_t^\eta$. Therefore, if every right-continuous L^2-martingale can be represented as a stochastic integral with respect to a Wiener process, then we can derive a stochastic differential equation for $E[f(\xi_t)|\Im_t^\eta]$. As is known, the central results of nonlinear filtering theory – the derivation of the stochastic equations, satisfied by the optimal nonlinear filter. Thus, the question of the stochastic integral representation of martingales is very important for filtering problems.

Therefore, the following question naturally arises: can any $\Im_t$-martingale be represented as a stochastic integral? It turned out that we have a positive answer to this question [1] when $\Im_t = \Im_t^w$, but in general this is not so. This is shown in the example of Kallianpur [2] (to whom M. Ior described it, and the latter, in turn, attributes the example to H. Kunita): let (w_t^1, w_t^2) be a Wiener process in R^2, and let $M_t = \int_0^t w_s^1 dw_s^2$. Choose $\Im_t = \Im_t^M$. Then $N_t = (w_t^1)^2 - t = 2\int_0^t w_s^1 dw_s^1$ is a $\Im_t^M$- martingale, but it cannot be represented as a stochastic integral with respect to M_t.

Indeed, on the one hand, it is clear that

$$\langle M\rangle_t = \int_0^t (w_s^1)^2 ds$$

is $\Im_t^M = \sigma\{M_s : 0 \le s \le t\}$-measurable and hence, $(w_t^1)^2$ is also $\Im_t^M$-measurable, because according to the mean value theorem, due to the continuity of the path of Wiener process,

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it is clear that for a.e. ω :

$$(w_t^1)^2 = \lim_{n\longrightarrow\infty} \frac{1}{t-(t-1/n)} \int_{t-1/n}^t (w_s^1)^2 ds = \lim_{n\longrightarrow\infty} \frac{\langle M\rangle_t - \langle M\rangle_{t-1/n}}{t-(t-1/n)}.$$

Therefore, $N_t = (w_t^1)^2 - t$ is a square integrable $\Im_t^M$- martingale.

Let us now assume that it can be represented as a stochastic integral $N_t = \int_0^t H_s dM_s$, where H_s is a predictable process for which $E\{\int_0^\infty H_s^2 d\langle M\rangle_s\} < \infty$. According to the Ito formula, we have $N_t = 2\int_0^t w_s^1 dw_s^1$ and hence

$$N_t = 2\int_0^t w_s^1 dw_s^1 = \int_0^t H_s dM_s = \int_0^t H_s w_s^1 dw_s^2.$$

From these relations follows the validity of the equalities (by virtue of independence $\langle w^1, w^2\rangle_t = 0$, therefore $\langle \int w^1 dw^1, \int w^1 dw^2\rangle_t = 0$):

$$0 = E\{2\int_0^t w_s^1 dw_s^1 - \int_0^t H_s w_s^1 dw_s^2\}^2 = 4E\int_0^t (w_s^1)^2 ds + E\int_0^t H_s^2 (w_s^1)^2 ds,$$

which, of course, is impossible.

On the other hand, in the 80s of the last century (Harrison and Pliska, 1981), it became clear that martingale representation theorems (along with Girsanov's absolutely continuous change of measure theorem) play an important role in modern financial mathematics. After Clark [1] obtained the formula for the stochastic integral representation for Brownian Motion functionals, many authors tried to explicitly find the integrand. The works of Haussmann (1979), Ocone [3], Ocone and Karatzas (1991), Karatzas, Ocone and Li (1991), Shyriaev and Yor (2003), Graversen, Shyriaev and Yor (2006) and Renaud and Remillard (2007) are especially important in this direction.

Hence, taking into account the needs of modern financial mathematics, it is not enough to know only the existence of an integral representation, it is necessary to be able to find the explicit form of the integrand of the integral representation. It is known that for stochastically smooth functionals, the integrand is calculated by Ocone's formula [3], which was later generalized by Glonti and Purtukhia [4], when only the filter of the functional is stochastically smooth. Here we study functionals whose filter is no longer smooth and propose a method for finding the integrand. We study the question of representing Brownian functionals as a stochastic Ito integral with an explicit form of the integrand. The considered class of functionals also includes functionals that are not smooth in the sense of Malliavin, to which both the well-known Clark-Ocone formula [3] and its generalization, the Glonti-Purtukhia formula [4], are inapplicable.

2 Martingale representation. Let a Brownian Motion $B = (B_t)$, $t \in [0,T]$, be given on a probability space $(\Omega, \Im, P)$, and let $\Im_t^B = \sigma\{B_u : 0 \le u \le t\}$.

Let $C_p^\infty(R^n)$ be the set of all infinitely differentiable functions $f : R^n \longrightarrow R$ such that f and all its partial derivatives have polynomial growth. Denote by Sm the class of smooth random variables F of the form $F = f(B_{t_1}, B_{t_2}, \dots, B_{t_n})$, $f \in C_p^\infty(R^n)$, $t_i \in [0,T]$.

Definition 1. The stochastic derivative (derivative in the Malliavin sense) of a smooth random variable F is defined as a random process D_tF defined by the relation [5]

$$D_tF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(B_{t_1}, B_{t_2}, \dots, B_{t_n}) I_{[0,t_i]}(t).$$

D is closable as an operator from $L_2(\Omega)$ to $L_2(\Omega; L_2([0,T]))$. Denote its domain of definition by $D_{1,2}$. This means that $D_{1,2}$ is equal to the closure of the class of smooth random variables in the norm

$$||F||_{1,2} := \{E[F^2] + E[||DF||^2_{L_2([0,T])}]\}^{1/2}.$$

Theorem 1. *Let $f(\cdot,\cdot) : [0,T] \times R^1 \longrightarrow R^1$ be a measurable bounded function, then the function $V(t,x) = E[\int_t^T f(s, B_s(\omega))ds | B_t = x]$ satisfies the requirements of the Ito formula and the the following stochastic integral representation is fulfilled*

$$\int_0^T f(s, B_s)ds = \int_0^T Ef(s, B_s)ds + \int_0^T V_x'(s, B_s)dB_s \quad (P - a.s.). \tag{1}$$

Proof. According to the well-known properties of the conditional mathematical expectation and the Markov property of Brownian Motion, we have

$$E\Big[\int_t^T f(s, B_s(\omega))ds | B_t\Big] = \int_t^T E[f(s, B_s(\omega))|B_t]ds$$

$$= \int_t^T E[f(s, B_s(\omega))|\Im_t^B]ds = \frac{1}{2(s-t)} \int_t^T f(s,y) \exp\{-\frac{(B_t - y)^2}{2(s-t)}\}dyds.$$

Therefore, it is easy to see that the function $V(t,x)$ satisfies the requirements of the Itô formula, so we can write

$$V(t, B_t) = V(0, B_0) + \int_0^t [V_s'(s, B_s) + \frac{1}{2}V_{xx}''(s, B_s)]ds$$

$$+ \int_0^t V_x'(s, B_s)dB_s \quad (P - a.s.). \tag{2}$$

On the other hand, it is obvious that

$$V(t, B_t) = E\Big[\int_t^T f(s, B_s(\omega))ds | \Im_t^B\Big] \quad (P - a.s.)$$

and hence the process

$$\int_0^t f(s, B_s(\omega))ds + V(t, B_t) = E\Big[\int_0^T f(s, B_s(\omega))ds | \Im_t^B\Big] := M_t$$

is a continuous martingale.

On the other hand, a continuous martingale of bounded variation starting from 0 is identically equal to 0 (see Theorem 39 [6]). Therefore, in equality (2.2) the term of bounded variation plus the additional term ($\int_0^t f(s, B_s(\omega))ds$) of bounded variation of the martingale M is equal to zero and, taking into account the equality

$$M_0 = V(0, B_0) = E\Big[\int_0^T f(s, B_s(\omega))ds|B_0\Big]$$

$$= E\Big[\int_0^T f(s, B_s(\omega))ds|\Im_0^B\Big] = \int_0^T Ef(s, B_s)ds \quad (P - a.s.)$$

the proof of the theorem is easily completed. □

Corollary. The following stochastic integral representation is valid

$$I_{\{B_T \le c\}} = \varPhi\Big(\frac{c}{\sqrt{T}}\Big) - \int_0^T \frac{1}{\sqrt{T-s}}\varphi\Big(\frac{c - B_s}{\sqrt{T-s}}\Big)dB_s \quad (P - a.s.),$$

where $\varPhi$ is the standard normal distribution and φ is its density function.

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Author(s) address(es):

Vakhtang Jaoshvili
Vladimir Komarov Tbilisi School of Physics and Mathematics N199
Vazha Pshavela Ave. 49, 0186 Tbilisi, Georgia
E-mail: vakhtangi.jaoshvili@gmail.com

Valeriane Jokhadze
Faculty of Economics and Business of I. Javakhishvili Tbilisi State University
University str. 13, 0186 Tbilisi, Georgia
E-mail: jokhadze.valeriane@gmail.com

Omar Purtukhia
Department of Mathematics and A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University
University str. 13, 0186 Tbilisi, Georgia
E-mail: o.purtukhia@gmail.com; omar.purtukhia@tsu.ge