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ON THE SYSTEM OF MAXWELL'S NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS *

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Abstract. Two one-dimensional models based on Maxwell's well-known system of non-linear partial differential equations are considered. Three schemes of construing exact solutions are stated. Some properties of the corresponding initial-boundary value problems are studied. Finite-difference scheme is built and its convergence is given.

Keywords and phrases: Nonlinear partial differential equations, Maxwell's system, unique solvability, exact solutions, finite-difference scheme, convergence.

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One-dimensional, two models based on Maxwell's system [1] of nonlinear partial differential equations (SNPDE) are considered.

Numerous scientific works are devoted to research of the SNPDE. One of such type model, is the well-known system of Maxwell's equations. In the quasi-stationary approximation, this system has the form [1]:

$$\frac{\partial H}{\partial t} = -rot\left(v_m rot H\right), \tag{1}$$

$$\frac{\partial \theta}{\partial t} = v_m \left(rot H\right)^2, \tag{2}$$

where $H = (H_1, H_2, H_3)$ is a vector of the magnetic field, θ is temperature, v_m characterizes the electro-conductivity of the substance. As a rule, this coefficient is function of argument θ. Equations (1) describe the process of diffusion of the magnetic field and equation (2) change of the temperature at the expense of Joule's heating.

System (1), (2) does not take into account many physical effects. For a more thorough description, first of all it is desirable to take into consideration heat conductivity. In this case together with (1), instead of (2) the following equation is considered [1]

$$\frac{\partial \theta}{\partial t} = v_m \left(rot H\right)^2 + div\left(k_m grad \theta\right), \tag{3}$$

where $k_m = k_m(\theta)$ is the coefficient of heat conductivity.

Note that, system (1), (2) can be reduced to an integro-differential form [2]. Many works are devoted to the investigation and numerical resolution of initial-boundary value

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problems for (1), (2) and (1), (3) systems and for integro-differential models corresponding to (1), (2) (see, for example, [2] - [12] and references therein).

For most SNPDEs, it is very difficult to find exact solutions and there is no general solution available in a close form. Nevertheless, it is known that the exact solution for SNPDEs is possible to construct in some particular cases.

Before attempting to solve a problem involving a PDE we would like to know if a solution exists, and, if it exists, if the solution is unique. Besides, in problem involving time, whether a solution exists $t > 0$ (global existence) or only up to a given value of t, i.e. only for $0 < t < t_0$ (finite time blow-up, shock formation).

Our aim is to continue to study one-dimensional versions of systems (1), (2) and (1), (3) in the case of the two-component magnetic field.

In the domain $Q = (0;1) \times (0;\infty)$, let us consider the following initial-boundary value problem for Maxwell's type one-dimensional (1), (2) system:

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left(S^\alpha \frac{\partial U}{\partial x}\right), \quad \frac{\partial V}{\partial t} = \frac{\partial}{\partial x}\left(S^\alpha \frac{\partial V}{\partial x}\right), \tag{4}$$

$$\frac{\partial S}{\partial t} = S^\alpha \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right], \tag{5}$$

$$U(0,t) = V(0,t) = 0, \quad U(1,t) = \psi_1 > 0, \quad V(1,t) = \psi_2 > 0, \tag{6}$$

$$U(x,0) = U_0(x), \quad V(x,0) = V_0(x), \quad S(x,0) = S_0(x) > 0. \tag{7}$$

Here $\alpha \in R; \psi_1, \psi_2$ are constants, and $U_0(x), V_0(x), S_0(x)$ are the given smooth functions.

It is easy to check that if $U_0(x) = \psi_1 x, V_0(x) = \psi_2 x$ and $S_0(x) = S_0 = const > 0$, then when $\alpha \neq 1$ the solution of problem (4) - (7) is:

$$U(,tx) = \psi_1 x, \quad V(x,t) = \psi_2 x, \quad S(x,t) = \left[S_0^{1-\alpha} + (1-\alpha)(\psi_1^2 + \psi_2^2)\right]^{\frac{1}{1-\alpha}}. \tag{8}$$

As can be seen from (8), for a finite value of time, namely, when $t_0 = S_0^{1-\alpha}/[(\psi_1^2 + \psi_2^2)(\alpha - 1))]$ and $\alpha > 1$, then the function $S(x,t)$ is not bounded. The above example shows that (4) - (7) has no global solution at all. So, the solution of problem (4) - (7) with smooth initial and boundary conditions can be blown up at a finite time. The questions of unique solvability of some cases of these type problems are studied in the abovementioned literature and in the number of other works as well. Using [2] it is not difficult to prove the following statement.

Theorem 1. *If $|\alpha| \leq 1/2$, then problem (4) - (7) has a unique solution.*

Note that if we add to (6) the following boundary conditions:

$$\left.\frac{\partial S}{\partial x}\right|_{x=0} = \left.\frac{\partial S}{\partial x}\right|_{x=1} = 0, \tag{9}$$

then U, V and S defined by formulas (8) are also solutions of the system with equations (4) and

$$\frac{\partial S}{\partial t} = S^{\alpha}\left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] + \frac{\partial^2 S}{\partial x^2}, \tag{10}$$

with (6), (7), (9) boundary and initial conditions. We conclude that for $\alpha > 1$, problem (4), (6), (7), (9), (10) has no global solution either.

We used some exact solutions constructed by the following schemes for the systems (4), (5) and (5), (10). First, the traveling wave solutions is sought: $U = U(x - \xi t)$, $V = V(x - \xi t)$, $S = S(x - \xi t)$ where ξ is the speed of the wave. Two type exact solutions for one-component magnetic field models (4), (5) and (4), (10) are also constructed. One of such solution has the form: $U = U(S)$ and $S = S(\partial U/\partial x)$ The second one is the solution found by the separation of variables $U = w(x)p(t), V = g(x)q(t)$.

If we introduce the following notation $E = S^{1/2}$ the problem (4) - (7) will be formulated as follows:

$$\frac{\partial U}{\partial t} - \frac{\partial}{\partial x}\left(E^{2\alpha}\frac{\partial U}{\partial x}\right) = 0, \quad \frac{\partial V}{\partial t} - \frac{\partial}{\partial x}\left(E^{2\alpha}\frac{\partial V}{\partial x}\right) = 0, \tag{11}$$

$$\frac{\partial E}{\partial t} = \frac{1}{2}E^{2\alpha-1}\left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right], \tag{12}$$

$$U(0,t) = V(0,t) = 0, \quad U(1,t) = \psi_1, \quad V(1,t) = \psi_2, \tag{13}$$

$$U(x,0) = U_0(x), \quad V(x,0) = V_0(x), \quad E(x,0) = [S_0(x)]^{1/2}. \tag{14}$$

Let us construct grids on $[0,1] \times [0,T]$. Divide the intervals $[0,1]$ and $[0,T]$ into equal M and J parts respectively, and introduce the following notations [13]:

$$h = 1/M, \quad \tau = T/J, \quad x_i = ih, \quad t_j = j\tau, \quad u_i^j = u(x_i, t_j),$$

$$\overline{\omega}_h = \{x_i = ih, i = 0,1,...,M\}, \quad \omega_h^* = \{x_i = (i - 1/2)\,h, i = 0,1,...,M\},$$

$$\omega_\tau = \{\tau_j = j\tau, i = 0,1,...,M\}, \quad \omega_{h\tau} = \overline{\omega}_h \times \omega_\tau \quad \omega_{h\tau}^* = \overline{\omega}_h^* \times \omega_\tau,$$

$$u_x = \frac{u_{i+1} - u_i}{h}, \quad u_{\overline{x}} = \frac{u_i - u_{i-1}}{h}, \quad u_i = u_i^{j+1}, \quad u_t = \frac{u_{i+1} - u_i^j}{\tau}.$$

Using usual notation and technique of building the finite difference schemes (see, for example, [13]) let us construct an implicit finite difference scheme for problem (11)-(14):

$$u_t^j = \left(e^{2\alpha}u_{\overline{x}}\right)_x, \quad v_t^j = \left(e^{2\alpha}v_{\overline{x}}\right)_x, \tag{15}$$

$$e_t^j = \frac{1}{2}e^{2\alpha-1}\left(u_{\overline{x}}^2 + v_{\overline{x}}^2\right), \tag{16}$$

$$u_0^j = v_0^j = 0, \quad u_M^j = \psi_1, \quad v_M^j = \psi_2, \tag{17}$$

$$u_i^0 = U_0(x_i), \quad v_i^0 = V_0(x_i), \quad e_i^0 = [S_0(x_{i+1/2})]^{1/2}, \quad i = 0,1,...,M-1, \tag{18}$$

where functions u and v are defined on the grid $\overline{\omega}_{h\tau}$ and function e is defined on the grid $\omega_{h\tau}^*$. Here unindexed values mean that the grid functions are taken at the point (x_i, t_{j+1}) or $(x_{i-1/2}, t_{j+1})$.

The approximations of the (15) - (18) scheme on the smooth solutions of problem (11) - (14) are of order $O(\tau + h^2)$. The following statement takes place [12].

Theorem 2. *If* $|\alpha| \leq 1/2$*, then the scheme* (15)-(18) *converges to a smooth solution of problem* (11) - (14) *in the grid functions space* L_2 *and the order of convergence is* $O(\tau + h^2)$*.*

Statements analogical to the abovementioned Theorems 1, 2 are true for the second problem (4), (6), (7), (9), (10) too.

The obtained results are a continuation of some studies of the papers [2]-[4], [12].

The proof of the theorems presented in this article, for wider nonlinearity, the description of algorithms for the approximate solution of the discussed problems, and the results of the corresponding numerical experiments are planned in the following article.

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