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MATHEMATICAL MODEL FOR THE PROTO-KARTVELIAN POPULATION DYNAMICS

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Abstract. This work discusses the first period (from L to XXX-XXV centuries BC) when the entire population spoke one Proto-Kartvelian language. This period is described by a Pearl-Verhulst-type mathematical model. For the unknown function of variable coefficients in the general case, which determines the number of Proto-Kartvelian speaking population at a given moment in time, the exact analytical solution of the Cauchy problem for the Bernoulli equation is found in quadratures. Analytical solutions have been found for specific values of coefficients (constant; qualitative; exponential; trigonometric functions) and the results have been analyzed. The generalization of the mathematical model is also discussed when in the area where the population lives, the assimilation of the neighboring tribes or the unnatural reduction of the population as a result of clashes with the neighboring tribes takes place. In this case, for the unknown function, the Cauchy problem is obtained for the Riccati-type equation. In some particular cases, when the inhomogeneous term has a special form, an analytical solution to the Riccati equation has been found.

Keywords and phrases: Proto-Kartvelian population dynamics, mathematical model, Riccati-type equation.

AMS subject classification (2010): 93A30, 00A71, 97M10, 97M70.

1 Introduction. According to Homer and Diodorus Siculus, the Pelasgians lived in Bronze Age Crete. Hecateos Miletus says that “before the Hellenes, the Peloponnese peninsula was inhabited by barbarians”. According to Herodotus, “the Pelasgians spoke a barbarous non-Greek language... the Pelasgian tribe forgot its old language when it became Hellenic.” The Pelasgian tribes spoke the Proto-Kartvelian language and were primarily found in the Peloponnese peninsula, Crete, and other islands, in ancient Asia and the Caucasus. At a certain stage of development (XXX-XXV centuries BC), the Pelasgian tribes experienced strong harassment from nomadic Indo-European and Semitic warlike tribes, as a result of which their territory of residence was significantly reduced and the population was divided into three parts over time: Proto-Svan, Speaking the Colchian-Georgian language, with the corresponding area of residence, and the third part, speaking Proto-Kartvelian, dispersed on the European continent. Subsequently, the process of dividing the Proto-Kartvelian language continued and today it is represented in the form of four languages: Georgian, Megrelian, Laz, and Svan. From a historical point of view, we consider its mathematical modeling as an innovative approach to describe the area of the population speaking the Proto-Kartvelian language and the process of language transformation, determining the number of the population speaking the relevant language in each period of time. We will assume that the formation of the hypothetical Common Kartvelian language began approximately no later than 7000 years ago. The territory of Proto-Kartvelian must have been the mountainous regions of the western and central part of the Little Caucasus where contacts with Proto-Indo-European and Proto-Semitic could have taken place [1].

1. To study the dynamics of the Proto-Kartvelian population speaking only one Proto-Kartvelian language from L to XXX-XXV centuries BC, the following nonlinear mathematical model has been proposed, described by an equation of the Riccati equation type. This mathematical model was built using the information obtained from linguistics, genetics, and archeology.

$$\begin{cases} \dfrac{du}{dt} = \alpha_1(t)u - \beta_1(t)u^2 - \gamma_1(t) \\ u(t)|_{t=0} = U_0 > 0 \end{cases}, \quad (1)$$

here $\alpha_1(t) \in C[0;T_1], \alpha_1(t) > 0$ demographic factor (natural co-indicator of reproduction and mortality); $\beta_1(t) \in C[0;T_1], \beta_1(t) > 0$ self-limiting factor; $\gamma_1(t) \in C[0;T_1]$ unnatural factor of population change (assimilation of other peoples by the Proto-Kartvelian population or on the contrary) and mortality due to wars; $u(t)$ is the unknown function (number of Proto-Kartvelian population at time t) and t is the independent variable (LXXX-XXV centuries BC), $t = 0$ is taken as the starting time (L century BC), $t = T_1$ (XXX-XXV centuries BC), $[0;T_1]$ - period of consideration (adequacy) of mathematical model, $u \in C^1[0;T_1], u(t) > 0; U_0$ is 1 million.

Let $u_p(t)$ be a particular solution of the (1) is known. We use the transformation $u(t) = v(t) + u_p(t)$, or the method is to find the general solution of (1), where $v(t)$ is an unknown new function. The analytic solution of (1) is

$$u(t) = \frac{exp[\int_0^t [\alpha_1(s) - 2\beta_1(s)u_p(s)]ds]}{\frac{1}{U_0 - U_{p0}} + \int_0^t \beta_1(\tau) exp[\int_0^\tau [\alpha_1(s) - 2\beta_1(s)u_p(s)]ds]d\tau} + u_p(t). \quad (2)$$

1.1 If $\gamma_1 \equiv 0$, then (1) is a particular quadratic case of Bernoulli's equation

$$\begin{cases} \frac{du}{dt} = \alpha_1(t)u - \beta_1(t)u^2 \\ u(t)|_{t=0} = U_0 > 0 \end{cases}. \quad (3)$$

The exact solution to equation (3) is

$$u(t) = \frac{exp[\int_0^t \alpha_1(s)ds]}{\frac{1}{U_0} + \int_0^t \beta_1(\tau) exp[\int_0^\tau \alpha_1(s)ds]d\tau} \quad (4)$$

1.2 If $\alpha_1(t) = \alpha_{10} = const$, $\beta_1(t) = \beta_{10} = const$, then the analytic solution of the problem (3) is given by the population ('S-shaped') function

$$u(t) = \frac{\frac{\alpha_{10}}{\beta_{10}}}{1 + (\frac{\alpha_{10}}{\beta_{10}U_0} - 1)exp[-\alpha_{10}t]}, \quad (5)$$

$$t = \frac{1}{\alpha_{10}} \ln \frac{[\frac{\alpha_{10}}{\beta_{10}} - U_0]u(t)}{U_0[\frac{\alpha_{10}}{\beta_{10}} - u(t)]}, \tag{6}$$

where $\lim_{t\to+\infty} u(t) = \frac{\alpha_{10}}{\beta_{10}}$. This shows that $\frac{\alpha_{10}}{\beta_{10}}$ is the limiting value of $u(t)$, i.e., the highest value that the population can reach given infinite time. Here, $\frac{\alpha_{10}}{\beta_{10}}$ is the saturation or ceiling value of the S-shaped curve. It is called the 'carrying capacity' of the population.

If $t = t^*$ and $u(t^*) = \frac{\alpha_{10}}{2\beta_{10}}$, then $u^{'}(t^*) = \frac{a_{10}^2}{4\beta_{10}}$, and the solution t^* of the equation $u^{''}(t) = 0$ is given by the S-shaped growth curve with the point of inflection $t^* = \frac{1}{\alpha_{10}} ln(\frac{\alpha_{10}}{\beta_{10}U_0} - 1)$.

The graph of the function (5) is called the population curves, as shown in Fig. 1.

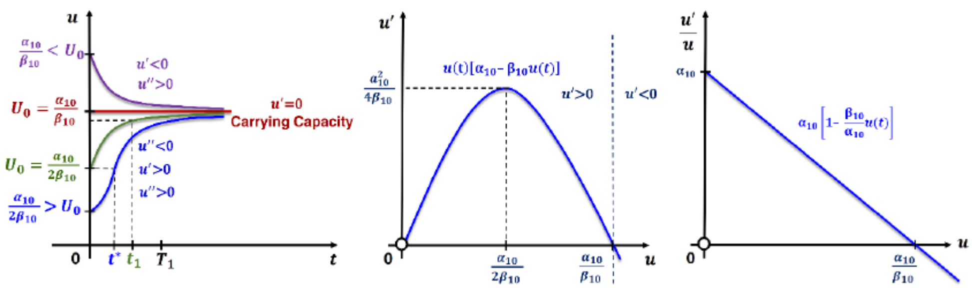

Figure 1:

2. Consider the case of a change in the population, including due to unnatural factors (assimilation of other neighboring peoples by the Proto-Kartvelian population or on the contrary) and mortality due to wars.

If $\alpha_1(t) = \alpha_{10} = const, \beta_1(t) = \beta_{10} = const, \gamma_1(t) = \gamma_{10} = const$ in (1), then $u_p(t) = d = const, D = \alpha_{10}^2 - 4\beta_{10}\gamma_{10} \geqslant 0, d_{1,2} = \frac{\alpha_{10} \pm \sqrt{\alpha_{10}^2 - 4\beta_{10}\gamma_{10}}}{2\beta_{10}}$.

a. $D = 0, u_p(t) = d = \frac{\alpha_{10}}{2\beta_{10}}, u(t) = \frac{4\beta_{10}U_0 + (2\beta_{10}U_0 - \alpha_{10})\alpha_{10}t}{2\beta_{10}[2 + (2\beta_{10}U_0 - \alpha_{10})t]}$.

b. $D > 0, u_p(t) = d_{1,2} = \frac{\alpha_{10} \pm \sqrt{\alpha_{10}^2 - 4\beta_{10}\gamma_{10}}}{2\beta_{10}}$,

$$u_{1,2}(t) = \frac{1}{\beta_{10}} \left\{ \frac{\left[2\beta_{10}U_0 - \alpha_{10} \mp \sqrt{\alpha_{10}^2 - 4\beta_{10}\gamma_{10}}\right] \sqrt{\alpha_{10}^2 - 4\beta_{10}\gamma_{10}} exp[\mp\sqrt{\alpha_{10}^2 - 4\beta_{10}\gamma_{10}}]t}{2\sqrt{\alpha_{10}^2 - 4\beta_{10}\gamma_{10}} \mp \left[2\beta_{10}U_0 - \alpha_{10} \mp \sqrt{\alpha_{10}^2 - 4\beta_{10}\gamma_{10}}\right] \left[exp[\mp\sqrt{\alpha_{10}^2 - 4\beta_{10}\gamma_{10}}t] - 1\right]} + \frac{\alpha_{10} \pm \sqrt{\alpha_{10}^2 - 4\beta_{10}\gamma_{10}}}{2} \right\}. \tag{7}$$

The graph of the function (7) is shown in Fig. 2.

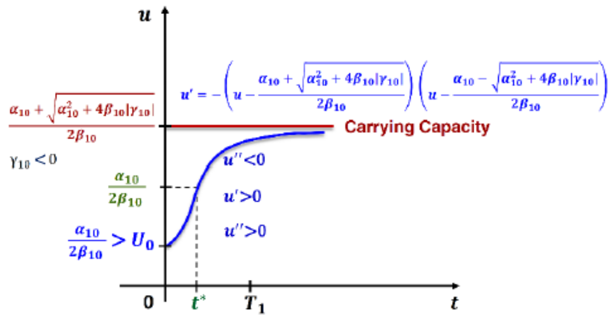

Figure 2:

3. If $\alpha_1(t) = \alpha_{10} = const, \beta_1(t) = \beta_{10} = const, \gamma_1(t) = \gamma_{10}exp[2\gamma_{20}t], \gamma_{10} < 0$ in (1), then $u_p(t) = \pm\sqrt{-\frac{\gamma_{10}}{\beta_{10}}}exp[\alpha_{10}t]$,

$$u_{1,2}(t) = \frac{exp[\alpha_{10}t \mp \frac{2\sqrt{-\beta_{10}\gamma_{10}}}{\alpha_{10}}(exp[\alpha_{10}t] - 1)]}{\frac{1}{U_0 \mp \sqrt{-\frac{\gamma_{10}}{\beta_{10}}}} + \beta_{10}\int_0^t exp[\alpha_{10}\tau \mp \frac{2\sqrt{-\beta_{10}\gamma_{10}}}{\alpha_{10}}(exp[\alpha_{10}\tau] - 1)]d\tau}$$

$$\pm\sqrt{-\frac{\gamma_{10}}{\beta_{10}}}exp[\alpha_{10}t].$$

4. If $\alpha_1(t) = 2e^{2t}, \beta_1(t) = e^t, \gamma_1(t) = e^t - 3e^{3t}$ in (1), then $u_p(t) = -e^t$,

$$u(t) = \frac{e^{2(e^{2t}-1)}}{\frac{1}{U_0+1} + \int_0^t e^{\tau+2(e^{2\tau}-1)}d\tau} - e^t.$$

Conclusion. In conclusion, we note that the proposed mathematical model to describe the dynamics of the Proto-Kartvelian population from L to XXX-XXV centuries BC, shows that the Proto-Kartvelian population in L century BC approximately 1 million grew to approximately 3 million by XXX-XXV centuries BC.

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