Reports of Enlarged Sessions of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 36, 2022

CONSISTENT ESTIMATOR OF PARAMETER IN THE HILBERT SPACE OF MEASURES

Zurab Zerakidze Omar Purtukhia

Abstract. Statistical structures in the Hilbert space of measures are considered and necessary and sufficient conditions for the existence of consistent estimators of the parameters are given.

Keywords and phrases: Consistent estimators, orthogonal statistical structure, weakly separable statistical structure, strongly separable statistical structure.

AMS subject classification (2010): 62H05, 62H12.

1 Introduction.

Let (E, S) be a measurable space with a given family of probability measures $\{\mu_i, i \in I\}$. We recall some definitions from [3]-[5].

Definition 1.1. An object $\{E, S, \mu_i, i \in I\}$ called a statistical structure.

Definition 1.2. A statistical structure $\{E, S, \mu_i, i \in I\}$ is called orthogonal (singular) if a family of probability measures $\{\mu_i, i \in I\}$ consists of pairwise singular measures (i.e. $\mu_i \perp \mu_j$, $\forall i \neq j$).

Definition 1.3. A statistical structure $\{E, S, \mu_i, i \in I\}$ is called weakly separable if there exists a family of S-measurable sets $\{X_i, i \in I\}$ such that $\mu_i(X_j) = \begin{cases} 1, & if\ i = j; \\ 0, & if\ i \neq j. \end{cases}$

Definition 1.4. A statistical structure $\{E, S, \mu_i, i \in I\}$ is called strongly separable if there exists a disjoint family of S-measurable sets $\{X_i, i \in I\}$ such that $\cup_{i \in I} X_i = E$ and $\mu_i(X_i) = 1,\ \ \forall i \in I$.

Let I be the set of parameters and let $B(I)$ be a σ-algebra of subsets of I which contains all finite subsets of I.

Definition 1.5. We will say that the statistical structure $\{E, S, \mu_i, i \in I\}$ admits a consistent estimator of parameter $i \in I$ if there exists at least one measurable mapping $\delta\ :\ (E, S)\ \longrightarrow\ (I, B(I))$, such that $\mu_i(\{x : \delta(x) = i\}) = 1,\ \ \forall\ i \in I$.

Remark 1.1. Strong separability implies weak separability, and weak separability implies orthogonality, but not vice versa.

Remark 1.2. If the statistical structure $\{E, S, \mu_i, i \in I\}$ admits a consistent estimator of parameter then this statistical structure is strongly separable, but not vice versa.

2 The consistent estimators of statistical structures

Theorem 1. *(see [6]) Let $\{E, S, \mu_i, i \in N\}$ $(N = \{1, 2, ...\})$ be an orthogonal statistical structure, then this statistical structure is strongly separable.*

Theorem 2. *Let $\{E, S, \mu_i, i \in N\}$ ($N = \{1, 2, ...\}$) be an orthogonal statistical structure, then this statistical structure admits a consistent estimator of parameter.*

Proof. Due to the singularity of statistical structure $\{E, S, \mu_i, i \in N\}$ ($N = \{1, 2, ...\}$) there exists the family of S-measurable sets $\{X_{ik}\}$ such that for any $i \neq k : \mu_i(X_{ik}) = 0$ and $\mu_i(E \setminus X_{ik}) = 0$. Therefore, if we consider the sets $X_i = \cup_{k \neq i}(E \setminus X_{ik})$, we get $\mu_i(X_i) = 0$. Hence, $\mu_i(E \setminus X_i) = 1$.

On the other hand, for $k \neq i$ we have $\mu_k(E \setminus X_i) = 0$. It means that the statistical structure $\{E, S, \mu_i, i \in N\}$ is weakly separable. Therefore, there exists the family of S-measurable sets $\{X_i, i \in N\}$ such that $\mu_i(X_j) = \begin{cases} 1, & if\ i = j; \\ 0, & if\ i \neq j \end{cases} \quad (i, j \in N)$.

Consider now the sets $\overline{X}_i = X_i \setminus (X_i \cap (\cup_{k \neq i} X_k))$, $i \in N$. It is obvious that these sets are S-measurable disjoint sets and $\mu_i(X_i) = 1$, $i \in N$. Let us define the mapping $\delta : (E, S) \longrightarrow (I, B(I))$ in the following way: $\delta(\overline{X}_i) = i$, $i \in N$.

Then we have $\{x : \delta(x) = i\} = \overline{X}_i$ and $\mu_i(\overline{X}_i) = \mu_i(\{x : \delta(x) = i\}) = 1$, $\forall i \in N$. Hence, δ is a consistent estimator of the parameter $i \in N$. □

3 The consistent estimators of statistical structures in the Hilbert space of measures. Let $\{\mu_i, i \in I\}$ be a family of probability measures on the space (E, S). For each $i \in I$ we denote by $\overline{\mu}_i$ the completion of the measure μ_i, and by $dom(\overline{\mu}_i)$ we denote the σ-algebra of all $\overline{\mu}_i$-measurable subsets of E. Let $S_1 = \cap_{i \in I} dom(\overline{\mu}_i)$.

Definition 3.1. A statistical structure $\{E, S_1, \overline{\mu}_i, i \in I\}$ is called strongly separable if there exists a family of S_1-measurable sets $\{Z_i, i \in I\}$ such that the relations are fulfilled: 1) $\mu_i(Z_i) = 1$, $\forall i \in I$; 2) $Z_{i_1} \cap Z_{i_2} = \emptyset$, $\forall i_1 \neq i_2$, $i_1, i_2 \in I$; 3) $\cup_{i \in I} Z_i = E$.

Definition 3.2. We will say that the orthogonal statistical structure $\{E, S_1, \overline{\mu}_i, i \in I\}$ admits a consistent estimator of the parameter $i \in I$ if there exists at least one measurable mapping $\delta : (E, S_1) \longrightarrow (I, B(I))$, such that

$$\overline{\mu}_i(\{x : \delta(x) = i\}) = 1, \ \forall\, i \in I.$$

Theorem 3. *Let $M_H = \oplus_{i \in I} H_2(\overline{\mu}_i)$ be a Hilbert space of measures, let E be a complete metric space, whose topological weights are not measurable in a wider sense. In order for the Borel orthogonal statistical structure $\{E, S_1, \overline{\mu}_i, i \in I\}$ to admit a consistent estimator of the parameter $i \in I$ in the theory of (ZFC) & (MA) it is necessary and sufficient that the correspondence $f \longleftrightarrow \psi_f$ defined by the equality*

$$\int_E f(x)\overline{\mu}_i(dx) = (\psi_f, \overline{\mu}_i) = l_f, \ \ \forall \overline{\mu}_i \in M_H$$

was one-to-one (here $f \in F(M_H)$).

Proof. **Sufficiency.** Since for each $f \in F(M_H)$ and $\overline{\mu}_i \in M_H$ the integral $\int_E f(x)\overline{\mu}_i(dx)$ is defined, then there exists a countable subset I_f in I for which $\int_E f(x)\overline{\mu}_i(dx) = 0$, $if\ i \notin$

I_f; $\sum\limits_{i \in I_f} \int\limits_E |f(x)|^2 \overline{\mu}_i(dx) < \infty$ and for any countable subset $\widetilde{I} \subset I$ and for the measure $\nu(C) = \sum\limits_{i \in \widetilde{I}} \int\limits_C g_i(x)\overline{\mu}_i(dx)$ we conclude that $\int_E f(x)\overline{\mu}_i(dx) = (\psi_f, \overline{\mu}_i), \quad \forall \overline{\mu}_i \in M_H$, is one-to-one. Therefore, the statistical structure $\{E, S_1, \overline{\mu}_i, i \in I\}$ is weakly separable. Consequently, There is a family of S_1-measurable sets $X_i, i \in I$, for which the following condition is satisfied: $\mu_{i_1}(X_{i_2}) = \begin{cases} 1, & if\ i_1 = i_2; \\ 0, & if\ i_1 \neq i_2. \end{cases}$

Further, we represent $\{\overline{\mu}_i, i \in I\}$, as an inductive sequence $\{\overline{\mu}_i < \omega_1\}$, where ω_1 denotes the first ordinal number of the power of the set I.

We define ω_1 sequence Z_h of parts of the space E such that the following relations hold: 1) Z_i is a Borel subset of E, $\forall i < \omega_1$; 2) $Z_i \subset X_i$, $\forall i < \omega_1$; 3) $Z_i \cap Z_{i'} = \emptyset$ for all $i < \omega_1,\ i' < \omega_1,\ i \neq i'$; 4) $\overline{\mu}_i(Z_i) = 1$, $\forall i < \omega_1$.

Suppose that $Z_{i_0} = X_{i_0}$. Suppose further that the partial sequence $\{Z_{i'}\}_{i'<i}$ is already defined for $i < \omega_1$. It is clear that $\overline{\mu}_i(\cup_{i'<i} Z_{i'}) = 0$. Thus there exists a Borel subset Y_i of the space E such that the following relations are valid: $\cup_{i'<i} Z_{i'} \subset Y_i \ \ and \ \ \overline{\mu}_i(Y_i) = 0$.

Assuming that $Z_i = X_i \setminus Y_i$, we construct the ω_1 sequence $\{Z_i\}_{i<\omega_1}$ of disjunctive measurable subsets of the space E. Therefore $\overline{\mu}_i(Z_i) = 1$ for all $i < \omega_1$ and the statistical structure $\{E, S_1, \overline{\mu}_i, i \in I\}$, $card I = c$, is strongly separable because there exists a family of elements of the σ-algebra $S_1 = \cap_{i \in I} dom(\overline{\mu}_i)$ such that: 1) $\overline{\mu}_i(Z_i) = 1, \quad \forall i \in I$; 2) $Z_i \cap Z_{i'} = \emptyset$ for all different i and i' from I; 3) $\cup_{i \in I} Z_i = E$.

For $x \in E$, we put $\delta(x) = i$, where i is the unique hypothesis from the set I for which $x \in Z_i$. The existence of such a unique hypothesis from I can be proved using conditions 2), 3).

Now let $Y \in B(I)$. Then $\{x : \delta(x) \in Y\} = \cup_{i \in Y} Z_i$. We must show that $\{x : \delta(x) \in Y\} \in dom(\overline{\mu}_i)$ for each $i \in I$.

If $i_0 \in Y$, then $\{x : \delta(x) \in Y\} = \cup_{i \in Y} Z_i = Z_{i_0} \cup (\cup_{i \in Y \setminus \{i_0\}} Z_i)$.

On the one hand, from the validity of condition 1), 2), 3) it follows that $Z_{i_0} \in S_1 = \cap_{i \in I} dom(\overline{\mu}_i) \subseteq dom(\overline{\mu}_{i_0})$.

On the other hand, the validity of the condition $\cup_{i \in Y \setminus \{i_0\}} Z_i \subseteq (E \setminus Z_{h_0})$ implies that $\overline{\mu}_{i_0}(\cup_{i \in Y \setminus \{i_0\}} Z_i) = 0$.

The last equality yields that $\cup_{i \in Y \setminus \{i_0\}} Z_i \in dom(\overline{\mu}_{i_0})$.

Since $dom(\overline{\mu}_{i_0})$ is a σ-algebra, we deduce that $\{x : \delta(x) \in Y\} = Z_{i_0} \cup (\cup_{i \in Y \setminus \{i_0\}} Z_i) \in dom(\overline{\mu}_{i_0})$.

If $h_0 \notin Y$, then $\{x : \delta(x) \in Y\} = \cup_{i \in Y} Z_i \subseteq (E \setminus Z_{i_0})$ and we conclude that $\overline{\mu}_{i_0}\{x : \delta(x) \in Y\} = 0$. The last relation implies that $\{x : \delta(x) \in Y\} \in dom(\overline{\mu}_{i_0})$.

Thus we have shown the validity of the relation $\{x : \delta(x) \in Y\} \in dom(\overline{\mu}_{i_0})$ for an arbitrary $i_0 \in I$. Hence, $\{x : \delta(x) \in Y\} \in \cap_{h \in H} dom(\overline{\mu}_h) = S_1$.

We have shown that the map $\delta : (E, S_1) \longrightarrow (I, B(I))$ is a measurable map. Since $B(I)$ contains all singletons of I we ascertain that $\overline{\mu}_i(\{x : \delta(x) = i\}) = \overline{\mu}_i(Z_i) = 1, \ \forall i \in I$.

Necessity. The existence of a consistent estimator of the parameter $\delta : (E, S_1) \longrightarrow (I, B(I))$ implies that $\overline{\mu}_i(\{x : \delta(x) = i\}) = 1$, $\forall\ i \in I$. Setting $X_i = \{x : \delta(x) = i\}$ for

$i \in I$ we get: 1) $\overline{\mu}_i(X_i) = 1$, $\forall i \in I$; 2) $X_{i_1} \cap X_{i_2} = \emptyset$ for all different parameters i_1 and i_2 from I; 3) $\cup_{i \in I} X_i = \{x : \delta(x) \in I\} = E$.

Therefore the statistical structure $\{E, S_1, \overline{\mu}_i, i \in I\}$ is strongly separable, hence, there exist S_1-measurable sets X_i $(i \in I)$, such that $\mu_i(X_{i'}) = \begin{cases} 1, & if\ i = i'; \\ 0, & if\ i \neq i'. \end{cases}$

Next, we associate the measure $\overline{\mu}_i$ with the function $I_{X_i}(x) \in F(M_H)$, the measure $\overline{\mu}_{i_1}$ – with the function $f_{i_1}(x) = f_1(x) I_{X_{i_1}}(x) \in F(M_H)$ and the measure $\nu(C) = \sum\limits_{i \in I_1 \subset I} \int\limits_C g_i(x) \overline{\mu}_i(dx) \in M_H$ – with the function $f(x) = \sum\limits_{i \in I_1} g_i(x) I_{X_i}(x) \in F(M_H)$. Then for the measure $\nu_1(C) = \sum\limits_{i \in I_2 \subset I} \int\limits_C g_i^1(x) \overline{\mu}_i(dx) \in M_H$ we have $\int\limits_E f(x) \nu_1(dx) = (\nu, \nu_1)$.

The above correspondence connects some function $f \in F(M_H)$ in correspondence with each linear continious functional l_f. If we identify functions in $F(M_H)$ that coincide with respect to measures $\{\overline{\mu}_i, i \in I\}$, then the correspondence will be bijective. $\square$

REFERENCES

1. Borovkov, A. A. Mathematical Statistics. Estimation of Parameters. Testing of Hypotheses. Nauka, Moscow, 1984; English transl., *Mathematical statistics. Gordon & Breach, Amsterdam*, 1998.
2. Ibramkhalilov, I., Skorokhod, A. Consistent Estimates of Parameters of Random Processes (Russian). *Naukova Dumka, Kiev*, 1980.
3. Zerakidze, Z. On consistent estimators for families of probability measures. *5-th Japan-USSR Symposium on Probability Theory and Mathematical Statistics, Kyoto University*, (1986), 62-63.
4. Zerakidze, Z. Construction of Statistical Structures. *Theory of probability and application.* **XV**, 3 (1986), 573-578.
5. Zerakidze, Z. Hilbert space of measures. *Ukrainian. Math. J.*. **38**, 2 (1986), 147-153.
6. Zerakidze, Z., Purtukhia, O. The weakly consistent, strongly consistent and consistent estimates of the parameters. *Reports of Enlarged Session of the Seminar of I. Vekua Institute of Applied Mathematics*, **31** (2017), 151-154.

Received 05.05.2022; revised 29.07.2022; accepted 25.09.2022.

Author(s) address(es):

Zurab Zerakidze
Gori State Teaching University
Chavchavadze Ave. 53, 1400 Gori, Georgia
E-mail: zura.zerakidze@mail.ru

Omar Purtukhia
Department of Mathematics, Faculty of Exact and Natural Sciences
A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University
University str. 13, 0186 Tbilisi, Georgia
E-mail: o.purtukhia@gmail.com; omar.purtukhia@tsu.ge