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POSITIVE INTEGERS REPRESENTED BY SOME BINARY FORMS

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Abstract. The formulae for the average number of representations of positive integers by a genus of positive binary quadratic forms are given. It gives us the opportunity to characterize all the primes or the primes multiplied by natural powers of 2 which can be represented by some binary quadratic forms.

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1 Introduction. Let $r(n,t)$ denote a number of representations of a positive integer n by a positive quadratic form $f = ax^2 + bxy + cy^2$. If the genus of the quadratic form f contains one class, then according to Newman Hall's theorem [1], the problem of obtaining exact formulas for $r(n;f)$ is solved completely. It follows from the results of [2] and [3] that half of "the sum $\rho(n,f)$ of a generalized singular series" that corresponds to a binary quadratic form f is equal to the average number of representations of a positive integer n by the genus containing this quadratic form. In particular, if a quadratic form belongs to a one-class genus, then for positive integer n

$$r(n;f) = \frac{1}{2}\rho(n;f). \tag{1}$$

The function $\rho(n;f)$ can be calculated as follows (see, [4], pp. 79, 80).

Theorem 1. *Let $f = ax^2 + bxy + cy^2$ be a primitive positive binary quadratic form with discriminant $d = b^2 - 4ac$, $(a,d) = 1$, $\Delta = -\frac{d}{4}$ if $2 \mid b$, $\Delta = -d$ if $2 \nmid b$; $\Delta = r^2\omega$ (ω is a square free number), $n = 2^\alpha m$ $(2 \nmid m)$, $\Delta = 2^\gamma \Delta_1$ $2 \nmid \Delta_1$, $p^l \| \Delta, p^\beta \| n$, $u = \prod\limits_{\substack{p\mid n\\ p\mid 2\Delta}} p^\beta$, then*

$$\rho(n;f) = \frac{\pi\chi_2 \prod\limits_{p\mid\Delta, p>2} \chi_p \sum\limits_{\nu\mid u} (\frac{-\Delta}{\nu})}{\Delta^{\frac{1}{2}} \prod\limits_{p\mid r, p>2} (1 - (\frac{-\omega}{p})\frac{1}{p}) L(1,-\omega)}.$$

Here

$$\chi_p = \Big(1 + \Big(\frac{p^{-\beta na}}{p}\Big)\Big) p^{\frac{1}{2}\beta} \ \text{ if } \ l \geq \beta + 1, \ \ 2 \mid \beta;$$

$$= \Big(1-\Big(\frac{-p^{-l}\Delta}{p}\Big)\frac{1}{p}\Big)\Big\{1+\Big(1+\Big(\frac{-p^{-l}\Delta}{p}\Big)\Big)\frac{\beta-l}{2}\Big\}p^{\frac{1}{2}l} \ \text{ if } \ l\le\beta,\ \ 2\mid l,\ \ 2\mid\beta;$$

$$= \Big(1-\Big(\frac{-p^{-l}\Delta}{p}\Big)\frac{1}{p}\Big)\Big(1+\Big(\frac{-p^{-l}\Delta}{p}\Big)\Big)\frac{\beta-l+1}{2}p^{\frac{1}{2}l} \ \text{ if } \ l\le\beta,\ \ 2\mid l,\ \ 2\nmid\beta;$$

$$= \Big(1+\Big(\frac{p^{-l}\Delta}{p}\Big)^{\beta+1}\Big(\frac{p^{-(\beta+l)}na\Delta}{p}\Big)\Big)p^{\frac{1}{2}(l-1)} \ \text{ if } \ l\le\beta,\ \ 2\nmid l;$$

$$= 0 \ \text{ if } \ l\ge\beta+1,\ \ 2\nmid\beta;$$

for $2\mid d$,

$$\chi_2 = 2^{\frac{\alpha}{2}+2} \ \text{ if } \ 2\mid\gamma,\ \ 0\le\alpha\le\gamma-3,\ \ 2\mid\alpha,\ \ m\equiv a \pmod 8);$$

$$= 0 \ \text{ if } \ 2\mid\gamma,\ \ 0\le\alpha\le\gamma-3,\ \ 2\mid\alpha,\ \ 2\mid\gamma,\ \ m\not\equiv a \pmod 8; \ \text{ or } \ 0\le\alpha\le\gamma-1,\ \ 2\nmid\alpha;$$

$$= \big(1+(-1)^{\frac{1}{2}(m-\alpha)}\big)2^{\frac{\alpha}{2}} \ \text{ if } \ 2\mid\gamma,\ \ \alpha=\gamma-2;$$

$$= \big(1+(-1)^{\frac{1}{2}(m-\alpha)}\big)2^{\frac{\gamma}{2}} \ \text{ if } \ 2\mid\gamma,\ \ \alpha\ge\gamma,\ \ 2\mid\alpha,\ \ \Delta_1\equiv 1 \pmod 4;$$

$$= 2^{\frac{\gamma}{2}} \ \text{ if } \ 2\mid\gamma,\ \ \alpha=\gamma,\ \ \Delta_1\equiv -1 \pmod 4;$$

$$= \big(2-(-1)^{\frac{1}{4}(\Delta_1+1)}\big)2^{\frac{\gamma}{2}}+\big(1+(-1)^{\frac{1}{4}(\Delta_1+1)}\big)(\alpha-\gamma-2)2^{\frac{\gamma}{2}-1}$$
$$\text{if } \ 2\mid\gamma,\ \ \alpha>\gamma;\ \ 2\mid\alpha,\ \ \Delta_1\equiv -1 \pmod 4;$$

$$= \big(1+(-1)^{\frac{1}{4}(\Delta_1-1)+\frac{1}{2}(m-a)}\big)2^{\frac{\gamma}{2}} \ \text{ if } \ 2\mid\gamma,\ \ \alpha\ge\gamma+1,\ \ 2\nmid\alpha,\ \ \Delta_1\equiv 1 \pmod 4;$$

$$= \big(1+(-1)^{\frac{1}{4}(\Delta_1+1)}\big)(\alpha-\gamma-1)2^{\frac{\gamma}{2}-1} \ \text{ if } \ 2\mid\gamma,\ \ \alpha\ge\gamma+1,\ \ 2\nmid\alpha,\ \ \Delta_1\equiv -1 \pmod 4;$$

$$= 2^{\frac{\alpha}{2}+2} \ \text{ if } \ 0\le\alpha\le\gamma-3,\ \ 2\nmid\gamma,\ \ 2\mid\alpha,\ \ m\equiv a \pmod 8;$$

$$= 0 \ \text{ if } \ 0\le\alpha\le\gamma-3,\ \ 2\nmid\gamma,\ \ 2\mid\alpha,\ \ m\not\equiv a \pmod 8 \ \text{ or } \ 0\le\alpha\le\gamma-2,\ \ 2\nmid\gamma,\ \ 2\nmid\alpha;$$

$$= \big(1+(-1)^{\frac{1}{4}(m-a)}\big)2^{\frac{1}{2}(\gamma-1)} \ \text{ if } \ 2\nmid\gamma,\ \ \alpha\ge\gamma-1,\ \ 2\mid\alpha,\ \ m\equiv a \pmod 4;$$

$$= \big(1+(-1)^{\frac{1}{4}(m+a)+\frac{1}{2}(m-\Delta_1 a)}\big)2^{\frac{1}{2}(\gamma-1)} \ \text{ if } \ 2\nmid\gamma,\ \ \alpha\ge\gamma-1,\ \ 2\mid\alpha,\ \ m\equiv -a \pmod 4;$$

$$= \big(1+(-1)^{\frac{1}{4}(m-\Delta_1 a)}\big)2^{\frac{1}{2}(\gamma-1)} \ \text{ if } \ 2\nmid\gamma,\ \ \alpha\ge\gamma,\ \ 2\nmid\alpha,\ \ m\equiv \Delta_1 a \pmod 4;$$

$$= \big(1+(-1)^{\frac{1}{4}(m+\Delta_1 a)+\frac{1}{2}(m-a)}\big)2^{\frac{1}{2}(\gamma-1)} \ \text{ if } \ 2\nmid\gamma,\ \ \alpha\ge\gamma,\ \ 2\nmid\alpha,\ \ m\equiv -\Delta_1 a \pmod 4;$$

for $2\nmid d$,

$$\chi_2 = 3 \ \text{ if } \ 2\mid\alpha,\ \ \Delta\equiv 3 \pmod 8;$$

$$= 0 \ \text{ if } \ 2\nmid\alpha,\ \ \Delta\equiv 3 \pmod 8;$$

$$= \alpha+1 \ \text{ if } \ \Delta\equiv 7 \pmod 8;$$

The values of $L(1;-\omega)$ can be calculated by the formulas of [5] (Theorem 15). Using the Theorem 1 we can characterize all the primes which can be represented by the binary form belonging to one-class genera. This is motivated by a problem in number theory that dates back at least to Fermat: for a given n, characterizing all the primes which can be written $p = x^2 + ny^2$ for some integers x, y. Euler studied the problem extensively, and was able to solve it for $n = 1, 2, 3, 4$, giving the first rigorous proofs of the following four theorems of Fermat (see, [6], 1–16]).

$$\begin{aligned} p =& x^2 + y^2 \text{ for some } x, y \in \mathbb{Z} \Leftrightarrow p = 2 \text{ or } p \equiv 1 (\mathrm{mod}\, 4),\\ p =& x^2 + 2y^2 \text{ for some } x, y \in \mathbb{Z} \Leftrightarrow p = 2 \text{ or } p \equiv 1, 3 (\mathrm{mod}\, 8),\\ p =& x^2 + 3y^2 \text{ for some } x, y \in \mathbb{Z} \Leftrightarrow p = 3 \text{ or } p \equiv 1 (\mathrm{mod}\, 3),\\ p =& x^2 + 4y^2 \text{ for some } x, y \in \mathbb{Z} \Leftrightarrow p \equiv 1 (\mathrm{mod}\, 4). \end{aligned}$$

The similar results in case of $n = 5, 6, 7$ are given in [6].

In this paper, we show how the formulas of Theorem 1 can be used to characterize all the primes represented by the binary forms belonging to one-class genera; we introduce this method in the case of binary forms of discriminants $d = -32, -36$ and -40.

In [7] we showed that the problem of obtaining formulas for the number of representations of numbers by binary forms belonging to multi-class genera can be easily reduced to the case of one-class genera. It gives us the opportunity to characterize all the products of the primes by the natural powers of the number 2 which can be represented by binary forms belonging to multi-class genera. This method is illustrated in the case of binary forms of the discriminant -44.

Basic results. The set of binary forms with the discriminant -32 splits into two genera, each consisting of one class with reduced forms respectively, $f_1 = x^2 + 8y^2$ and $f_2 = 3x^2 + 2xy + 3y^2$.

Furthermore, the set of binary forms with the discriminant -36 splits into two genera, each consisting of one class with reduced forms, respectively, $f_3 = x^2 + 9y^2$ and $f_4 = 2x^2 + 2xy + 5y^2$.

The set of binary forms of the discriminant -40 splits into two genera, each consisting of one class with reduced forms respectively, $f_5 = x^2 + 10y^2$ and $f_6 = 2x^2 + 5y^2$. The corresponding representation functions can be calculated by the formulas of Theorem 1. According to this theorem we have the following result.

Theorem 2. *a prime p is of the form $x^2 + 8y^2$ for some $x, y \in \mathbb{Z}$ if and only if $p \equiv 1 (\mathrm{mod}\, 8)$. A prime p is of the form $3x^2 + 2xy + 3y^2$ for some $x, y \in \mathbb{Z}$ if and only if $p \equiv 3 (\mathrm{mod}\, 8)$. A prime p is represented by $x^2 + 9y^2$ if and only if $p \equiv 1 (\mathrm{mod}\, 12)$. A prime p is represented by $2x^2 + 2xy + 5y^2$ if and only if $p \equiv 5 (\mathrm{mod}\, 12)$ or $p = 2$. A prime p is represented by $x^2 + 10y^2$ if and only if $p \equiv 1 (\mathrm{mod}\, 40)$ or $p \equiv 9 (\mathrm{mod}\, 40)$ or $p \equiv 11 (\mathrm{mod}\, 40)$ or $p \equiv 19 (\mathrm{mod}\, 40)$. A prime p is represented by $2x^2 + 5y^2$ if and only if $p \equiv 7 (\mathrm{mod}\, 40)$ or $p \equiv 13 (\mathrm{mod}\, 40)$ or $p \equiv 23 (\mathrm{mod}\, 40)$ or $p \equiv 37 (\mathrm{mod}\, 40)$ or $p = 5$ or $p = 2$.*

The set of binary forms with discriminant -44 forms one genus, which consists of three classes with reduced forms $f_8 = x^2 + 11y^2$, $f_9 = 3x^2 + 2xy + 4y^2$ and $f_{10} = 3x^2 - 2xy + 4y^2$. Let $n = 2^\alpha p$ ($\alpha \in N$, p is a prime number). Then, according to Theorem 2 of the paper [7], $r(2^\alpha p; f_8) = r(2^\alpha p; f_9) = r(2^\alpha p; f_{10}) = r(2^{\alpha-2} p; f)$, where $\alpha \geq 2$ and $f = x^2 + xy + 3y^2$ is a binary form belonging to a one-class genera.

It is clear, that

$$r(2^\alpha p; f_8) = r(2^\alpha; f_9) = 0, \text{ if } 2 \nmid \alpha.$$

It follows from Theorem 1, that for any number $2^{\alpha-2}p$ ($\alpha-2\geq 0$, $\alpha\in N$, $2 \mid (\alpha-2)$, p is a prime number), $r(2^{\alpha-2}p;f)\neq 0$ if and only if $(\frac{p}{11})=1$, or $p=11$.

The arguments above yield

Theorem 3. *For a given prime p and natural α a number $2^{\alpha}p$ is represented by any form of the discriminant -44 if and only if $2 \mid \alpha$ and $p\equiv 3,4,5,9 (\bmod 11)$ or $p=11$ and $2 \mid \alpha$.*

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