Reports of Enlarged Sessions of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 36, 2022

TO NUMERICAL REALIZATION OF HIERARCHICAL MODELS OF I.VEKUA

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Abstract. Problems of numerical realization of I. Vekua's model for the one-dim case are studied. For constructing the model, as is well known, the Legendre polynomial system is used. The same problem is solved too when the basis is another- full system that satisfies Neyman-type boundary conditions. In both cases algorithms are created by which constructing solutions are equal. In the general case, Vekua-type models have investigated the problems of satisfying the boundary conditions on the face of surfaces.

Keywords and phrases: Elastic thin-walled structures, problems of satisfying the boundary conditions (PSBC), refined theory in a wide sense.

AMS subject classification (2010): 30G20, 74B20, 74.130, 74K90.

The PSBC on the face of surfaces for elastic thin-walled structures has a sufficiently deep and long history and we retained the essential remarks of I. Vekua [1] with respect to these problems. He introduced new expressions of type [1, (7.2.c)] which are "coordinated with the boundary conditions (BC) on the lateral surfaces, which form the 2-dim boundary value problems. Then Vekua constructed approximate solutions of displacement vector u and stress tensor series of Legendre polynomial system which are not compatible with the boundary data on the face of surfaces $S^{\pm}$. These approaches may turn out to be rather rough values near the face of surfaces."[1,pg. 79] For refined theories in a wide sense [2, Ch.I] the PSBC are studied in very different kinds. For example, by E.Reissner [3] and S.Ambartsumian [4] models, a priori satisfied BC on $S^{\pm}$, but we underline [5, p.49] that these ones are artificial and obscure. We cited also the article [6], where BC on the face of surfaces is a priori satisfied exactly. It is evident that the way used in [1] represents the incompletely Bubnov-Galerkin method as the BVPs for elastic plates contain the equilibrium equations and BCs on the lateral faces S and surfaces $S^{\pm}$. The last BCs in [1] are neglected but they are used in constructing two-dim systems of partial linear differential equations (PDE) corresponding to the following 3-dim systems of PDEs [details 1, Ch.1, § 2]:

$$\frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}P^i\right)+\Phi=0. \tag{1}$$

Now, following [7, 2], for the stress vector acted on $S^{\pm}$ we use a priori assumptions:

$$P^3=\frac{1}{2}\left[\left(1+\frac{x^3}{h}\right)\vartheta^{(+)}g^{+}+\left(1-\frac{x^3}{h}\right)\vartheta^{(-)}g^{-}\right]+\sum_{k=1}^{\infty}\overset{k}{P}{}^3\left[p_{k+1}\left(\frac{x^3}{h}\right)-p_{k-1}\left(\frac{x^3}{h}\right)\right]. \tag{2}$$

Elastic plates with constant thickness (2) have such simple forms in linear as well in nonlinear cases:

$$T_3(x,y,z)=\frac{1}{2h}\left[(h+z)\,g^{+}+(h-z)\,g^{-}\right]+\sum_{k=1}^{\infty}\overset{k}{T}_3(x,y)\left[p_{k+1}(z/h)-p_{k-1}(z/h\right]. \qquad (3)$$

$$T_{i3}(x,y,\pm h)=\sigma_{i3}+\sigma_{i3}u_{i,3}=g_i^{\pm},(x,y,\pm h)\in S^{\pm}.$$

From (3) also follows the possible application to arbitrary model from to refined theories in the wide sense as well as to theories of [3,4]too without some artificial restrictions of type

$$\sigma_{33,3}(x,y,\pm\, h)=0\,,\;\; \sigma_{ij,j}=f_i,(x,y,\pm h)\in S^{\pm},\sigma_{ij}\in C^3(D(x,y)\times[-h,h]).$$

Then the usefulness of the Vekua-Rektorys-Galerkin type method for (3) and BCs on $S$with cutting BCs (3) we get approximate two-dim models for any integer N (details [2, Ch.II, 7].

To prove this fact we use the excellent examples of [8] and consider when (1) is a 1-dim elastic beam and let $u_\alpha=\varepsilon_{\alpha i}=\sigma_{\alpha i}=\Phi_\alpha=0,h=1,\sigma_{33}=(\lambda+2\mu)u_{3,3}$. Then we get the following BVP:

$$-u''(x)=f(x),-1<x<1,u'(-1)=\alpha,u'(1)=\beta. \qquad (4)$$

If $z(x)=u(x)-\frac{\alpha+\beta}{2}x-\frac{\beta-\alpha}{4}x^2+u_0$, the problem (4) is equivalent to the following one:

$$-z''(x)=f\,(x)+\frac{\beta-\alpha}{2},\; z'(-1)=z'(1)=0. \qquad (5)$$

Let us consider two systems:

$$\{p_n(x)\}_{n=0,1,...}\, p_n(\pm1)=(\pm1)^n-are\,Legerndre\,polynomials;$$

$$\begin{cases} q_0(x)=-1,\\ q_1(x)=-3\int_{-1}^{1}(x-t)p_1(t)dt-p_0=-\frac{1}{5}p_3(x)+\frac{6}{5}p_1(x),..,\\ q_n(x)=-(2n+1)\int_{-1}^{1}(x-t)p_n(t)dt=-\frac{p_{n+2}-p_n}{2n+3}+\frac{p_n-p_{n-2}}{2n-1},n=2,3,... \end{cases}.$$

$$q_n'(x)=-(p_{n+1}-p_{n-1}).$$

Now consider two examples.
Example 1.
Let us find the solution of (5) by the set $z(x)=\sum_{k=1}^{\infty}z_kq_k(x)$ (having a unique solution if$\int_{-1}^{1}z(x)dx=0,i.e.z_0=0$).Then by projective method we have [6]: $z(x)=\frac{1}{3}q_1(x)=-\frac{1}{15}p_3+\frac{6}{15}p_1,-z''=p_1,z'(\pm1)=0$.
Example 2.
Let us consider BVP (4), when $f(x)=p_1(x)$, $\alpha=\beta$. The solution of this BVP is denoted

as $v(x) = \sum_{n=1}^{\infty} v_n p_n(x) + u_0$. Then if we used the direct method from [6] with respect to Vekua system[1] for this case we have:$v(x) = \left(\frac{2}{5} + \alpha\right) p_1(x) - \frac{1}{15} p_3(x) + u_0$.

As we see, both solutions:$u(x) = z(x) + \alpha x + u_0 = \left(\frac{2}{5} + \alpha\right) p_1 - \frac{1}{15} p_3 + u_0 = v(x)$ are equal. The reason for the equality of these solutions is the following. It is true that the Legendre polynomial system is not presented as a basic system [8], really the first equation of the Vekua algebraic system was formed as a scalar product of equation (4) and $p_0 = 1$. But resulting expression represent the necessary and sufficient conditions for the existence of the unique (by to within of a constant summand) solution to Neyman's problem. Now we consider the case when in (4):$f(x) = \frac{1}{2} f_0 = const$, $\beta - \alpha + f_0 \neq 0$,i.e.the BVP (4) is unsolvability. In this case, let $u(x) = \sum_{n=0}^{\infty} u_n p_n(x)$. Then Vekua's system ($n \neq 0$) can be rewritten in the form:

$$(-u''(t), p_{2n}(x)) = 0, (n = 1, 2, ..).$$

$$\Rightarrow \sum_{k=0}^{n-1} k(2k+1) u_{2k} + n(2n+1) \sum_{k=n+1}^{\infty} u_{2k} = \frac{1}{2}(\beta - \alpha). \tag{6}$$

$$(-u''(t), p_{2n+1}(x)) = 0, (n = 0, 1, 2, ..).$$

$$\Rightarrow \sum_{k=0}^{n-1} (k+1)(2k+1) u_{2k+1} + (n+1)(2n+1) \sum_{k=n}^{\infty} u_{2k+1} = \frac{1}{2}(\alpha + \beta). \tag{7}$$

From (6) and (7) immediately follows:

$$\sum_{k=n+i}^{\infty} u_{2k} = 0 \Rightarrow u_2 = \frac{1}{2}(\beta - \alpha), u_{2k} = 0, (k = 2, 3, ...);$$

$$u_1 = \frac{1}{2}(\alpha + \beta), u_{2k+1} = 0, (k = 1, 2, ...),$$

$$u(x) = u_0 + \frac{1}{2}(\alpha + \beta) p_1(x) + \frac{1}{6}(\beta - \alpha) p_2(x).$$

$$-u''(x) = \frac{1}{6}(\alpha - \beta) p''(x) = \frac{1}{2}(\beta - \alpha); u'(-1) = \alpha . u'(1) = \beta.$$

At last it should be stressed that if $\alpha = \beta$, then we have

$$u(x) = u_0 + \alpha p_1(x) \ .i.e. \ - u'' = 0,$$

instead of $-u''(x) = 0.5 f_0$. We must remember the words [1, p. 52]:

The (7. 18h, i) is a strong elliptic system of PDEs for $N \geq 3$ "but we don't rewrite this one in a more expanded form and shall not deal with the investigation of problems of existence and uniqueness in the general form [1]".

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Received 12.05.2022; revised 18.07.2022; accepted 11.09.2022.

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