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ABOUT SOME PROPERTIES OF ONE SPECIAL CLASS OF THE GENERALIZED MÖBIUS-LISTING'S BODIES *

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Abstract. In this work Generalized Möbius-Listing hollow bodies $GML_m^n\{p,q\}$ are studied, whose radial cross section are Grandi Roses (or Rhodonea curves). The work is devoted to the study of the control parameters of Grandi Roses (p,q) and control parameters of the GML-bodies (m,n) on the corresponding 3-dimensional structure. In other words, the following question is studied: if a flat figure, bounded by a certain Grandi Rose and having a certain number of connected components, is a radial cross section of a $GML_m^n\{p,q\}$-body, then how many connected components does the corresponding of 3-dimensional geometric object display.

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1 Introduction.

In previous articles we have dealt with cylinders of various shapes. After connecting the ends we obtain GML_m^n bodies. We cut these GML_m^n bodies with "chordal or radial knives" - the "knife" was a straight line on the plane! We tried to answer the question of how many different objects would result from such an intersection! Now we consider hollow cylinders, whose cross sections are Grandi Roses (or Rhodonea curves), and after connecting their ends, we ask the following question: how many different fluids can be poured into this three-dimensional object without mixing? In this article, we use the term Grandi Rose throughout. In other words: If the radial cross-section of the GML_m^n bounded by the Grandi roses consists of a certain number of connected components, how many different connected components will the corresponding three-dimensional GML_m^n object have?

2 Notations and abbreviations.

Further we will use the notation accepted in earlier works [1-4]. In this article we use the following notations:

- X, Y, Z is the ordinary notation for space coordinates;
- ϱ and ψ is classic polar coordinates;

On the one hand we will rely on the analytical representation of the well-known Grandi roses in polar coordinates

$$\varrho = \rho \cos(\tfrac{p}{q}\psi) \tag{1}$$

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On the other hand, we will use the analytical representation of our generalized Möbius-Listing bodies (see f.e.formulas (2) and (6*) in [1]) , which was given in the earlier papers [1-3] and where the radial section is Grandi roses, respectively.

$$\begin{aligned} X(\varrho,\psi,\theta) &= [R+\rho\cos(\tfrac{p}{q}\psi)\cos(\psi+\tfrac{n\theta}{m})]\cos(\theta),\\ Y(\varrho,\psi,\theta) &= [R+\rho\cos(\tfrac{p}{q}\psi)\cos(\psi+\tfrac{n\theta}{m})]\sin(\theta),\\ Z(\varrho,\psi,\theta) &= \rho\cos(\tfrac{p}{q}\psi)\sin(\psi+\tfrac{n\theta}{m}). \end{aligned} \tag{2}$$

As can be seen from (2), the structure of each particular $GML_m^n\{p,q\}$ body depends on four parameters - p,q,m,n. Two parameters p, and $q\neq 0$ are arbitrary integer numbers and define the the structure of the petals of the particular Grandi rose (i.e. shape and $m(p,q)$ number of symmetry of the radial cross section of the corresponding GML_m^n-body). n is an arbitrary integer number, which defines character of twisting, of the $GML_m^n\{p,q\}$. If the number is $n>0$ then the twisting is clockwise and $n<0$ is counterclockwise [1].

3 Structure of $GML_m^n\{p,q\}$ hollow bodies. Proposal 1. If $p=0$, or $p/q=1$ (i.e. $p=q$), then Grandi rose is a circle and so m may be any natural number and also for any integer number n - corresponding $GML_m^n\{p,p\}$ hollow body can hold only one fluid - this body is a torus with a twisted surface and its radial cross section always is a circle.

Proposal 2. If $\frac{p}{q}=(2k+1)$ is an odd integer number, then corresponding Grandi rose has $(2k+1)$ pieces of "uncut leaf" (petals) that are symmetrically positioned relative to the center of symmetry, respectively in this case always $m=(2k+1)$ and:

A.) if $gcd(m,n)=m$, then the corresponding $GML_m^n\{p,q\}$ hollow body can hold $(2k+1)$ - different unmixed fluids;

B.) if $gcd(m,n)=1$, then the corresponding $GML_m^n\{p,q\}$ hollow body can hold only one fluid; This is an analogue of the Möbius phenomenon (after full **radial cutting** [4] we have one geometric object, although the index of symmetry m can be an odd number) for the 3-dimensional body;

C.) if $gcd(m,n)=j$, then corresponding $GML_m^n\{p,q\}$ hollow body can hold j - different unmixed fluids;

Proposal 3. If $\frac{p}{q}=(2k)$ is an even integer number, then the corresponding Grandi rose has $(4k)$ pieces of "uncut leaf" (petals) that are symmetrically positioned relative to the center of symmetry, respectively in this case always $m=(4k)$ and:

A.) if $gcd(m,n)=m$, then the corresponding $GML_{4k}^n\{p,q\}$ hollow body can hold $(4k)$ - different unmixed fluids;

B.) if $gcd(m,n)=1$, then the corresponding $GML_{4k}^n\{p,q\}$ hollow body can hold only one fluid; This is an analogue of the standard Möbius phenomenon (after full **"diametral"** cutting [4] we have one geometric object) for the 3-dimensional body;

C.) if $gcd(m,n)=j$, then the corresponding $GML_{4k}^n\{p,q\}$ hollow body can hold j -different unmixed fluids;

Proposal 4. If $\frac{p}{q}=\frac{2k+1}{2}$ is already a fraction, then the corresponding Grandi rose has $m=2(2k+1)$ symmetry with depth level-2 (i.e m pieces intersected petals, that are symmetrically positioned relative to the center of symmetry, respectively in this case always $m=2(2k+1)$ and:

A.) if $gcd(m,n)=m$, then the corresponding $GML^n_{4k+2}\{p,q\}$ hollow body can hold $2(2k+1)$ - different unmixed fluids;

B.) if $gcd(m,n)=1$, then the corresponding $GML^n_{4k+2}\{p,q\}$ hollow body can hold 2 -different unmixed fluids;

C.) if $gcd(m,n)=j$, then the corresponding $GML^n_{4k+2}\{p,q\}$ hollow body can hold $2j$ -different unmixed fluids;

Proposal 5. If $\frac{p}{q}=\frac{1}{2k}$ is fraction, then the corresponding Grandi rose has $m=2$ symmetry with depth level-$2k$ and:

A.) if n is even number, then the corresponding $GML^n_2\{p,q\}$ hollow body can hold $4k$ - different unmixed fluids;

B.) if n is odd number, then the corresponding $GML^n_2\{p,q\}$ hollow body can hold $2k$ - different unmixed fluids;

C.) in this case Möbius phenomenon does not exist;

Remark 1. When $k=1$, then this particular Grandi rose is called Dürer's folium.

Proposal 6. If $\frac{p}{q}=\frac{1}{2k+1}$ is a fraction, then the corresponding Grandi rose has $m=1$ symmetry with depth level-$k+1$ and:

A.) for all values of n, the corresponding $GML^n_1\{p,q\}$ hollow body can hold $k+1$ - different unmixed fluids;

The proofs are based on the analytical representation (2) and on the well-known answer to the question: how many independent cyclic subgroups does the finite permutation group contains (This fact has been successfully used to prove analogous results [4]). Experimental observation of the structure of the Grandi roses allows the following general results to be suggested:

Proposal 7* without proof.If $\frac{p}{q}$ is already a fraction,and

A.) If $p=(2k+1), k=1,2..$, and $q=2i, i=1,2..$, then such Grandi Rose has $m(p,q)=2p=2(2k+1)$ symmetry, with depth level- $q=2i$. In this case if $gdc(m,n)=j, j=1,2,,m$ then the corresponding $GML^n_{2p}\{p,q\}$ hollow body can hold $(j\times q)$ -different unmixed fluids;

B.) If $p=(2k), k=1,2..$, and $q=(2i+1), i=1,2..,$, then such Grandi Rose has $m(p,q)=2p=4k$ symmetry, with depth level- $q=2i+1$. In this case if $gdc(m,n)=j, j=1,2,,m$ then corresponding $GML^n_{2p}\{p,q\}$ hollow body can hold $(j\times q)$- different unmixed fluids;

C.) If $p=(2k+1), k=1,2..$, and $q=(2i+1), i=1,2..$,then such Grandi Rose has $m(p,q)=p=2k+1$ symmetry, with depth level- $q=i+1$. In this case if $gdc(m,n)=j, j=1,2,,m$ then the corresponding $GML^n_p\{p,q\}$ hollow body can hold $(j\times(i+1))$- different unmixed fluids;

All these cases have in common that if the represented number p/q is already a fraction, then the Möbius phenomenon does not occur for any value of the number n.

In Table 1 the values below the figures indicate how many different unmixed fluids the corresponding hollow GML bodies can hold.

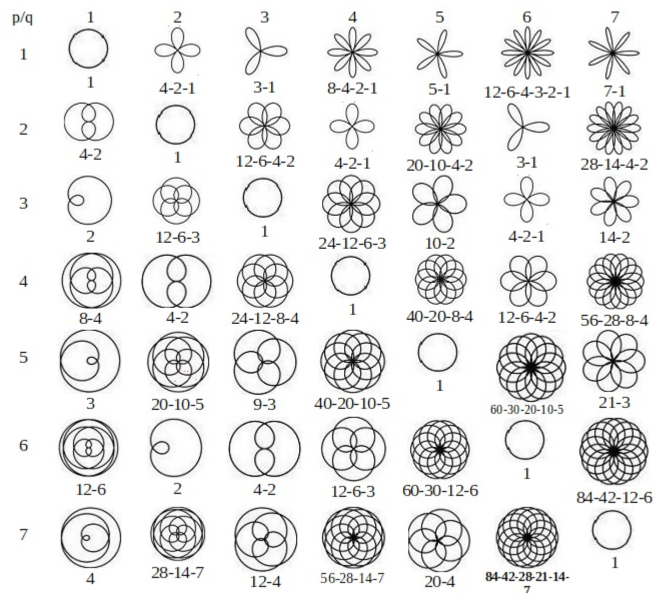

Figure 1: Grandi Roses with different p and q.

REFERENCES

1. Tavkhelidze I., Ricci P.E. Classification of a wide set of Geometric figures, surfaces and lines (Trajectories), *Rendiconti Accademia Nazionale delle Science detta dei XL, Memorie di Matematica e Applicazioni,* 124^o, XXX, 1 (2006), 191–212.
2. Tavkhelidze, I., Caratelli, D., Gielis, J., Ricci, P.E., Rogava, M., Transirico, M. On a Geometric Model of Bodies with “Complex” Configuration and Some Movements - *Modeling in Mathematics- Chapter 10* - Atlantis Transactions in Geometry 2, Springer (2017), 129-159.
3. Gielis, J., Tavkhelidze, I. The general case of cutting of Generalized Mbius-Listing surfaces and bodies - *4open 2020, vol.3, 2020. article N7 48 pages* www.4open-sciences.org https://doi.org/10.1051/fopen/2020007.
4. Tavkhelidze, I., Gielis, J. The process of cutting GML_m^n bodies with d_m- knives. *Reports of Enalged Session of the Seminar of I. Vekua Institute of Applied Mathematics*, **32** (2018), 67-70.
5. Ricci P.E. Grandi’s roses, Chebyshev polynomials and pseudo-Chebyshev functions (Georgian). *Scientific-popular journal “Matematika” of Tbilisi State University*, **7** (2021), 41-55.

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