Reports of Enlarged Sessions of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 36, 2022

THE PERIODIC PROBLEM FOR ONE CLASS OF FIRST ORDER HYPERBOLIC SYSTEMS

Irine Sigua Mariam Rashoian

Abstract. For one class of normally hyperbolic systems of the first order, a periodic problem with respect to a spatial variable is considered. The correctness of the problem is proved, and in some cases the solution of the periodic problem is written out explicitly.

Keywords and phrases: Normally hyperbolic systems, periodic problem, existence and uniqueness of a solution, explicit representation of the solution.

AMS subject classification (2010): 35F45, 35F46.

In the half-strip $D : 0 < x < l, t > 0$ of the plane Q_{xt} for a first-order linear system

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} + Bu = F(x,t),\ (x,t) \in D, \tag{1}$$

where A and B are n-order quadratic matrices, $F = (F_1, ..., F_n)$ is given, while $u = (u_1, ..., u_n)$ are unknown vector-functions, consider a periodic problem set as follows: in the domain D find a regular solution $u = (u_1, ..., u_n)$ to system (1) satisfying the initial condition

$$u(x,0) = \varphi(x),\ 0 \le x \le l, \tag{2}$$

and the periodicity condition

$$u(0,t) = u(l,t),\ t \ge 0, \tag{3}$$

with respect to the variable x.

Note that problem (1)-(3) is a special case of non-local problems. Nonlocal problems for partial differential equations and systems of various structures have been considered in numerous papers (see, for example, [1]-[9] and the literature cited therein).

Below we consider the case when (1) is a normally hyperbolic system: system (1) is called normally hyperbolic at a point (x,t) if the characteristic determinant $p(x,t,\lambda) = \det(\lambda I + A)$ of this system at this point with respect to the variable λ has only real roots $\lambda_1, ..., \lambda_l$ and at the same time

$$k_i = n - rank(\lambda_i I + A(x,t)),\ i = 1, ..., l, \tag{4}$$

where k_i is the multiplicity of the root λ_i, $i = 1, ..., l$, and I is the n-order unit matrix.

It should be noted that a strictly hyperbolic system, i.e., when all roots of the characteristic determinant $p(x,t,\lambda)$ are real and simple, is a special case of normally hyperbolic system.

Let us assume below that A is a constant matrix and system (1) is normally hyperbolic. Denote by V_i the kernel of the matrix operator $Q_i = \lambda_i I + A$, acting in the space R^n , whose dimension, by virtue of equality (4), is equal to k_i, and through $\nu_{ij},\ j = 1, ..., k_i$, we denote the basis of this kernel. Thus,

$$Q_i\nu_{ij} = (I + A)\nu_{ij} = 0,\ j = 1, ..., k_i,$$

and vectors $\nu_{i1}, ..., \nu_{ik_i}$ are linearly independent. Whence it follows that ν_{ij} is an eigenvector corresponding to the eigenvalue $-\lambda_i$.

It is proved that in the case of normal hyperbolicity of system (1) the square matrix K of order n whose columns consist of the vectors $\nu_{ij},\ i = 1, ..., l;\ j = 1, ..., k_i$, i.e., $K = (\nu_{11}...\nu_{1k_1}...\nu_l 1...\nu_{lk_l})$ is non-degenerate and the following important equality

$$K^{-1}AK = D_0 \tag{5}$$

holds, where D_0 is a diagonal matrix whose $(m_i + j)$-th element on the diagonal and $m_1 = 0, m_i = \sum_{s=1}^{i-1} k_i,\ i > 1, j = 1, ..., k_i$, is equal to the number $-\lambda_i$. i.e.

$$D_0 = diag(-\lambda_1, ..., -\lambda_1, ..., -\lambda_l, ..., -\lambda_l). \tag{6}$$

If we pass to a new unknown vector $\nu = (\nu_1, ..., \nu_n)$ according to the equality $u = K\nu$ and multiply from the left both parts of the system (1) by the matrix K^{-1}, then by virtue of (5) we get

$$\frac{\partial \nu}{\partial t} + D_0\frac{\partial \nu}{\partial x} + C\nu = F_1(x, t),\ (x, t) \in D, \tag{7}$$

where $C = K^{-1}BK,\ F_1 = K^{-1}F$.

Taking into account equalities (6), we rewrite system (7) in the form of separate scalar equations

$$\frac{\partial \nu_{m_i+j}}{\partial t} - \lambda_i\frac{\partial \nu_{m_i+j}}{\partial x} + \sum_{i=1}^{n} c_{m_i+j,s}\nu_s = F_{1,m_i+j}, \tag{8}$$

and conditions (2) and (3) will take the following form

$$\nu_{m_i+j}(x, 0) = \psi_{m_i+j}(x),\ 0 \le x \le l, \tag{9}$$

$$\nu_{m_i+j}(0, t) = \nu_{m_i+j}(l, t),\ t \ge 0, \tag{10}$$

$$i = 1, ..., l;\ j = 1, ..., k_i,$$

where $\psi = K^{-1}\varphi$.

Remark 1. Suppose $\lambda_i < 0$ for a fixed $i, 1 \le i \le l$. Divide the domain $D : 0 < x < l,\ t > 0$ into two parts $D_+ : D \cap \{x + \lambda_i t \ge 0\}$ and $D_- : D \cap \{x + \lambda_i t < 0\}$. It is easy to check that if the function $w(x, t)$ satisfies the equation $\frac{\partial w}{\partial t} - \lambda_i\frac{\partial w}{\partial x} = f(x, t)$ in the domain D, and the initial condition $w(x, t) = \omega(x)$ on the segment [0, l] of the axis Ox and the

boundary condition $w(0,t)=\mu(t)$ on the positive part of the Ot axis, then this function is given in the area D by the following formulas

$$w(x_0,t_0)=w(x_0+\lambda_i t_0)+\int_0^{t_0} f(x_0+\lambda_i t_0-\lambda_i t,t)dt,\ \ for\ P(x_0,t_0)\in \bar{D_+}, \tag{11}$$

$$w(x_0,t_0)=\mu(t_0+\frac{x_0}{\lambda_i})+\int_{t_0+\frac{x_0}{\lambda_i}}^{t_0} f(x_0+\lambda_i t_0-\lambda_i t,t)dt,\ \ for\ P(x_0,t_0)\in \bar{D_-}. \tag{12}$$

Similar formulas are valid when $\lambda_i>0$ and $\lambda_i=0$.

By integrating the equations (8) along the corresponding characteristics of the system (1), and taking into account formulas (11), (12), as well as initial conditions (9) and periodicity conditions (10), problem (1) - (3) is equivalently reduced to a system of Volterra - type integral equations which is uniquely solvable in the class $C^1(\bar{D})$.

Thus, the following theorem is true.

Theorem. Let A be a constant matrix and let system (1) be normally hyperbolic, $B\in C^1(\bar{D})$. Then for any vector functions $F\in C^1(\bar{D})$ and $\varphi\in C^1([0,l])$, where φ satisfies the matching condition $\varphi(0)=\varphi(l)$, there exists a unique regular solution $u\in C^1(\bar{D})$ of the problem (1) - (3).

Remark 2. Under the conditions of the above theorem, when $B=0$ and $F=0$, the unique regular solution $u\in C^1(\bar{D})$ of the problem (1) - (3) can be written explicitly:

$$u(x,t)=\sum_{i=1}^{l}\sum_{j=1}^{k_i}\nu_{m_i+j}(x,t)\nu_{ij}, \tag{13}$$

where

$$\nu_{m_i+j}(x,t)=\begin{cases}\psi_i((m-1)l+x+\lambda_i t),\\ when\ -\lambda_i t-(m-1)l<x<l,\ \frac{m-1}{-\lambda_i}l<t<\frac{m}{-\lambda_i}l,\\ \psi_i(ml+x+\lambda_i t),\\ when\ 0<x<-\lambda_i t-(m-1)l,\ \frac{m-1}{-\lambda_i}l<t<\frac{m}{-\lambda_i}l,\end{cases} \tag{14}$$

in the case $\lambda_i<0$, and

$$\nu_{m_i+j}(x,t)=\begin{cases}\psi_i(-(m-1)l+x+\lambda_i t),\\ when\ 0<x<ml-\lambda_i t,\ \frac{m-1}{\lambda_i}l<t<\frac{m}{\lambda_i}l,\\ \psi_i(-ml+x+\lambda_i t),\\ when\ ml-\lambda_i t<x<l,\ \frac{m-1}{\lambda_i}l<t<\frac{m}{\lambda_i}l,\end{cases} \tag{15}$$

in the case $\lambda_i>0$. Here in the formulas (11) - (13) : $\psi=K^{-1}\varphi$.

REFERENCES

1. Aizicovici, S., McKibben, M. Existence results for a class of abstract nonlocal Cauchy problems. *Nonlinear Anal. Ser. A: Theory Methods*, **39**, 5 (2000), 649-668.
2. Avalishvili, G., Gordeziani, D. On one class of spasial nonlocal problems for some hyperbolic equations. *Georgian Math. J.*, **7**, 3 (2000), 417-425.
3. Benedetti, I., Loi, N.V., Malaguti, L., Taddei, V. Nonlocal diffusion second order partial differential equations. J. Differential Equations **262**, 3 (2017), 1499-1523.
4. Bitsadze, A.V. Some Classes of Partial Differential Equations (Russian). *Nauka, Moscow*, 1981.
5. Bitsadze, A.V. On the theory of nonlocal boundary value problems (Russian). *Dokl. Akad. Nauk SSSR*, **777**, 1 (1984), 17-19.
6. Bitsadze, A.V., Samarskii, A.A. Some elementary genaralizations of linear boundary value problems (Russian). *Dokl. Akad. Nauk SSSR*, **185** (1969), 739- 740; *translation in Sov. Math., Dokl.*, **10** (1969), 398-400.
7. Cannon, J.R. The solution of the heat equation subject to the specification of energy. *Quart. Appl. Math.*, **21** (1963), 155-160.
8. Chang, J.-Ch. Existence and compactness of solutions to impulsive differential equations with nonlocal conditions. *Math. Methods Appl. Sci.*, **39**, 2 (2016), 317-327.
9. Gordeziani, D.G. Methods for Solving a Class of Nonlocal Boundary Value Problems (Russian). *Tbilisi. Gos. Univ., Inst. Prikl. Mat.*, 1981.

Received 24.05.2022; revised 30.08.2022; accepted 15.09.2022.

Author(s) address(es):

Irine Sigua
Georgian Technical University
M. Kostava str. 77, Tbilisi 0175, Georgia
E-mail: irinasigua@mail.ru

Mariam Rashoian
Georgian Technical University
M. Kostava str. 77, Tbilisi 0175, Georgia
E-mail: rashoian96@mail.ru