Reports of Enlarged Sessions of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 36, 2022

MARTINGALE REPRESENTATION OF ONE NON-SMOOTH FUNCTIONAL OF BROWNIAN MOTION

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Abstract. The well-known Clark-Ocone formula (1984) allows constructing a constructive martingale representation for stochastically smooth Brownian functionals. In the case of stochastically non-smooth functionals, if the conditional mathematical expectation of the functional with respect to the natural filtration of the Brownian motion is stochastically smooth, one can use the Glonti-Purtukhia generalization (2017) of the Clark-Ocone formula. Here we study functional to which the above results cannot be applied and derive a constructive martingale representation.

Keywords and phrases: Brownian functional, Malliavin (stochastic) derivative, martingale representation, Clark-Ocone formula, Glonti-Purtukhia representation.

AMS subject classification (2010): 60H07, 60H30.

1 Introduction. Martingale representation of Brownian functionals states that any square-integrable Brownian martingale is represented as a stochastic integral with respect to Brownian motion. The first proof of the martingale representation theorem was implicitly provided by Ito himself. Many other articles were written afterward on this problem and its applications but one of the pioneer work on explicit descriptions of the integrand is certainly the one by Clark ([1]). Those of Haussmann (1979), Ocone ([2]), Ocone and Karatzas (1991) and Karatzas, Ocone and Li (1991) were also particularly significant. A nice survey article on the problem of martingale representation was written by Davis (2005). In many papers using Malliavin calculus or some kind of differential calculus for stochastic processes, the results are quite general but unsatisfactory from the point of view of explicitness. Shiryaev and Yor (2003) proposed a method based on Ito's formula to find explicit martingale representations for functionals of the running maximum of Brownian motion.

Let B_t be a Brownian motion on a standard filtered probability space $(\Omega, \Im, \Im_t, P)$ and let $\Im_t = \Im_t^B$ be the augmentation of the filtration generated by B. One of the important properties of Ito's stochastic integral is the following: if f is a square integrable adapted process, then the process $\xi_t = \int_0^t f(s,\omega)dB_s(\omega)$ is a martingale with respect to the filtration $\{\Im_t\}_{t\geq 0}$. On the other hand, according to the well-known Clark formula, the inverse statement (so-called martingale representation theorem) is also true. The constructive martingale representation is based on the Malliavin (stochastic) derivative and in Brownian motion case it is known as the Clark-Ocone formula ([2]):

Theorem 1. *If F is differentiable in the Malliavin sense, $F \in D_{1,2}$*[1]*, then the integrand in the Clark representation is $E[D_t F|\Im_t^B]$, where D_t is the so called Malliavin stochastic derivative*[2] *of F.*

It has turned out that the requirement of smoothness of functional can be weakened by the requirement of smoothness only of its conditional mathematical expectation. We (with prof. O. Glonti, 2014) considered Brownian functionals which are not stochastically differentiable and established the method of finding of integrand (see [4]):

Theorem 2. *Assume that $g_t = E[F|\Im_t^B]$ is a Malliavin differentiable functional ($g_t(\cdot) \in D_{1,2}$) for almost all $t \in [0,T)$. Then the following stochastic integral representation is valid:*

$$g_T = F = EF + \int_0^T \nu_u dB_u \quad (P - a.s.), \tag{1}$$

where

$$\nu_u := \lim_{t \uparrow T} E[D_u g_t|\Im_u^B] \quad \text{in} \quad L_2([0,T] \times \Omega).$$

Obviously, there are functionals that do not satisfy even the weakened Glonti-Purtukhia condition. It should be noted that we are familiar with such functionals for which when taking the conditional mathematical expectation, a term similar to the original functional is separated from it, either as a term or as a factor. Such, for example, is the integral over the Lebesgue measure $\int_0^T u_s(\omega)ds$ of a process $u_s(\omega)$ that does not satisfy the Clark-Ocone condition, but satisfies the weakened Glonti-Purtukhia condition (here the conditional mathematical expectation gives an adapted to filtration term of the same type), and the corresponding representation for it was obtained in [5]. Another functional of this type is $I_{\{B_T^* \le x\}}$ ($B_T^* = max_{t \in [0,T]} B_t$), which we learned about from Andrei Ionescu (PhD student at King's College London), for which we are grateful (here the conditional mathematical expectation gives a multiplier of the same type adapted to filtration).

It is obvious that the mentioned last functional is not stochastically smooth [3] (therefore, we cannot use the Clark-Ocone formula) and we propose here a method for obtaining a constructive martingale representation[4]. In addition, even the conditional mathematical expectation of this functional is not stochastically smooth and, therefore, neither the Glonty-Purtukhia generalization of the Clark-Ocone formula (1) is applicable to it.

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[1] $D_{1,2}$ is equal to the closure of the class of smooth random variables (i.e. random variables F of the form $F = f(B_{t_1}, B_{t_2}, \ldots, B_{t_n})$, $f \in C_p^\infty(R^n)$, $h_i \in L_2([0,T])$) in the norm

$$||F||_{1,2} := \{E[F^2] + E[||DF||^2_{L_2([0,T])}]\}^{1/2}.$$

[2] The derivative of a smooth random variable $F = f(B_{t_1}, B_{t_2}, \ldots, B_{t_n})$ is the stochastic process given by $D_t F = \sum_{i=1}^n \frac{\partial}{\partial x_i} f(B_{t_1}, B_{t_2}, \ldots, B_{t_n}) I_{[0,t_i]}(t)$ (see Definition 1.2.1 [3]).

[3] The event indicator I_A is Malliavin differentiable if and only if the probability $P(A)$ equals zero or one (see Proposition 1.2.6 [3]).

[4] A different method of obtaining the corresponding representation was proposed by Andrei Ionescu in our personal correspondence.

2 Constructive martingale representation

Theorem 3. *The following stochastic integral representation is valid*

$$I_{\{B_T^* \le a\}} = P(B_T^* \le a) - 2\int_0^T I_{\{B_s^* \le a\}} \frac{1}{\sqrt{T-s}} \varphi\Big(\frac{a - B_s}{\sqrt{T-s}}\Big) dB_s \qquad (P - a.s),$$

where φ is the density function of the standard normal distribution.

Proof. We introduce the notation

$$f(t,x,y) = E(I_{\{B_T^* \le a\}} | B_t = x, B_t^* = y).$$

According to the Markov property of the two dimensional process (B_t, B_t^*) we can write

$$f(t, B_t, B_t^*) = E(I_{\{B_T^* \le a\}} | B_t, B_t^*) = E(I_{\{B_T^* \le a\}} | \Im_t^B).$$

Hence, due to the Levy's theorem, it is obvious that $f(t, B_t, B_t^*)$ is a martingale.

Further, it is not difficult to see that

$$f(t,x,y) = E(I_{\{B_T^* \le a\}} | B_t = x, B_t^* = y) = E(I_{\{B_{t,T}^* + B_t \le a\}} I_{\{B_t^* \le a\}} | B_t = x, B_t^* = y)$$

$$= I_{\{y \le a\}} E(I_{\{B_{t,T}^* + x \le a\}} | B_t = x, B_t^* = y) = I_{\{y \le a\}} E(I_{\{B_{t,T}^* \le a - x\}} | B_t = x) =$$

$$= I_{\{y \le a\}} P(B_{t,T}^* \le a - x) = I_{\{y \le a\}} \frac{1}{\sqrt{2\pi(T-t)}} \int_{-(a-x)}^{a-x} \exp\left\{-\frac{u^2}{2(T-t)}\right\} du.$$

It is clear that for all $y : f(\cdot, \cdot, y) \in C^{1,2}((0,T) \times R)$. Moreover, B_t^* is an increasing process. Therefore, according to the It formula, taking into account that any continuous martingale of finite variation is equal to a constant, for all $t < T$ we obtain that (P-a.s.)

$$f(t, B_t, B_t^*) = f(0, B_0, B_0^*) + \int_0^t f_x'(s, B_s, B_s^*) dB_s$$

or what is the same

$$E(I_{\{B_T^* \le a\}} | \Im_t^B) = P(B_T^* \le a) + \int_0^t f_x'(s, B_s, B_s^*) dB_s. \tag{2}$$

It is evident that

$$f_x'(t,x,y) = -I_{\{y \le a\}} \frac{2}{\sqrt{2\pi(T-t)}} \exp\left\{-\frac{(a-x)^2}{2(T-t)}\right\}.$$

Hence

$$f_x'(t, B_t, B_t^*) = -I_{\{B_t^* \le a\}} \frac{2}{\sqrt{2\pi(T-t)}} \exp\left\{-\frac{(a - B_t)^2}{2(T-t)}\right\}$$

and therefore (2) can be rewritten in the following form

$$E(I_{\{B_T^*\le a\}}|\Im_t^B) = P(B_T^*\le a) - 2\int_0^t I_{\{B_s^*\le a\}}\frac{1}{\sqrt{2\pi(T-s)}}\exp\left\{-\frac{(a-B_s)^2}{2(T-s)}\right\}dB_s$$
$$= P(B_T^*\le a) - 2\int_0^t I_{\{B_s^*\le a\}}\frac{1}{\sqrt{T-s}}\varphi\left(\frac{a-B_s}{\sqrt{T-s}}\right)dB_s.$$

Passing now to the limit in the last relation as $t \longrightarrow T$ (the limit of left-hand side exists by the Levy theorem and the limit of the right side - since the integrand is square integrable w. r. t. the Lebesgue measure), we obtain the stochastic integral representation

$$I_{\{B_T^*\le a\}} = P(B_T^*\le a) - 2\int_0^T I_{\{B_s^*\le a\}}\frac{1}{\sqrt{T-s}}\varphi\left(\frac{a-B_s}{\sqrt{T-s}}\right)dB_s.$$

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Received 10.05.2022; revised 20.07.2022; accepted 12.09.2022.

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