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ON THE ACCURACY OF A PROJECTION METHOD FOR A ONE-DIMENSIONAL NONLOCAL PARABOLIC EQUATION

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Abstract. The initial boundary value problem for a nonlinear diffusion model is considered. The Galerkin method is used for the approximation of the solution with respect to the spatial variable. The error of the method is estimated.

Keywords and phrases: Diffusion model, Galerkin method, error estimate.

AMS subject classification (2010): 35K55, 65M15, 65M60.

1 Statement of the problem. Let us consider the nonlinear differential equation with the nonlocal diffusion term

$$u_t - a\Big(\int_0^l u\,dx\Big)u_{xx} = f(x,t), \tag{1}$$
$$0<x<l, \quad 0<t<T, \quad a(\lambda)\ge const>0, \quad -\infty<\lambda<\infty,$$

and the initial boundary conditions

$$\begin{gathered} u(x,0)=\varphi(x), \quad 0<x<l,\\ u(0,t)=u(l,t)=0, \quad 0\le t<T. \end{gathered} \tag{2}$$

As to the background of the problem, note that in 1997 M. Chipot and B. Lovat [4] formulated for the first time the problem for the case $x\in R^n$, $n\ge 1$, analyzed its significance for application and investigated the problem of solution existence and uniqueness. In the works that were published subsequently, for the same and other similar classes of parabolic equations and systems with nonlocal terms such topics as solvability and solution uniqueness, asymptotic behavior of a solution were studied, proposed and investigated numerical algorithms and the results of numerical experiments were presented. In this connection, we refer to the works [1, 2, 3, 5, 6, 7] and the literature cited therein.

In the present paper, for the discretization of the problem (1), (2) with respect to the variable x the Galerkin method with sine-series expansion is used, also the approximation accuracy is studied.

2 Assumptions. Let us assume that there exists a sufficiently smooth solution of the problem (1), (2). Suppose a is a continuous function in R and $0<a_0\le a(z)<\infty$, $|a(z_1)-a(z_2)|\le L|z_1-z_2|$, $-\infty<z_1,z_2<\infty$. Let the representations

$$f(x,t)=\sum_{i=1}^{\infty} f_i(t)\sin\frac{i\pi x}{l}\,, \quad \varphi(x)=\sum_{i=1}^{\infty}\varphi_i\sin\frac{i\pi x}{l}\,,$$

$$f_i(t)=\frac{2}{l}\int\limits_0^l f(x,t)\sin\frac{i\pi x}{l}\,dx\,,\quad \varphi_i=\frac{2}{l}\int\limits_0^l \varphi(x)\sin\frac{i\pi x}{l}\,dx\,,\ \ i=1,2,\dots,$$

hold for the functions $f(x,t)$ and $\varphi(x)$, and for the solution $u(x,t)$ the expansion

$$u(x,t)=\sum_{i=1}^{\infty}u_i(t)\sin\frac{i\pi x}{l} \tag{3}$$

be valid, where $u_i(t)$ the coefficients satisfy the system of equations

$$u_i'+a\left(\frac{2l}{\pi}\sum_{j=1}^{\infty}\frac{1}{2j-1}\,u_{2j-1}\right)\left(\frac{i\pi}{l}\right)^2u_i=f_i(t),\quad i=1,2,\dots, \tag{4}$$

and the initial conditions

$$u_i(0)=\varphi_i,\quad i=1,2,\dots\ . \tag{5}$$

3 Method. The Galerkin method is meant to be used. The solution of the problem (1), (2) has the form of a finite sum

$$u_n(x,t)=\sum_{i=1}^{n}u_{ni}(t)\sin\frac{i\pi x}{l}\,, \tag{6}$$

the coefficients of which are defined from the system of ordinary differential equations

$$u_{ni}'+a\left(\frac{2l}{\pi}\sum_{j=1}^{n'}\frac{1}{2j-1}\,u_{n,2j-1}\right)\left(\frac{i\pi}{l}\right)^2u_{ni}=f_i(t),\quad i=1,2,\dots,n, \tag{7}$$

with the initial conditions

$$u_{ni}(0)=\varphi_i,\quad i=1,2,\dots,n. \tag{8}$$

Here $n'=[\frac{n+1}{2}]$ is the integer part of the value enclosed in square brackets.

4 Definition of the method error. Under the method error we understand the difference $\Delta u_n(x,t)=u_n(x,t)-p_nu(x,t)$, where $u_n(x,t)$ and $p_nu(x,t)$ are defined by the formula (6) and the formula

$$p_nu(x,t)=\sum_{i=1}^{n_i}u_i(t)\sin\frac{i\pi x}{l}\,. \tag{9}$$

As follows from (3), $p_nu(x,t)$ is the principal part of the exact solution.

From (6) and (9) we conclude that the equality $\Delta u_n(x,t) = \sum_{i=1}^{n} \Delta u_{ni}(t) \sin \frac{i\pi x}{l}$ is valid, where $\Delta u_{ni}(t) = u_{ni}(t) - u_i(t)$, $i = 1,2,\ldots,n$. Let us write a system of equations for $\Delta u_{ni}(t)$, $i = 1,2,\ldots,n$. For this we use the equalities (4), (5), (7), (8) and take into account that $AB - CD = \frac{1}{2}\left[(A-C)(B+D) + (A+C)(B-D)\right]$. As a result using $i = 1,2,\ldots,n$ we obtain the sought system of equations

$$\begin{aligned}
\big(\Delta u_{ni}(t)\big)' + \frac{1}{2}\Bigg\{\Bigg[a\bigg(\frac{2l}{\pi}\sum_{j=1}^{n'}\frac{1}{2j-1}\,u_{n,2j-1}(t)\bigg)& \\
- a\bigg(\frac{2l}{\pi}\sum_{j=1}^{n'}\frac{1}{2j-1}\,u_{2j-1}(t)\bigg)\Bigg]\bigg(\frac{i\pi}{l}\bigg)^2\big(u_{ni}(t)+u_i(t)\big)& \\
+\Bigg[a\bigg(\frac{2l}{\pi}\sum_{j=1}^{n'}\frac{1}{2j-1}\,u_{n,2j-1}(t)\bigg) + a\bigg(\frac{2l}{\pi}\sum_{j=1}^{n'}\frac{1}{2j-1}\,u_{2j-1}(t)\bigg)\Bigg]\bigg(\frac{i\pi}{l}\bigg)^2\Delta u_{ni}(t)\Bigg\} &= \psi_i(t),
\end{aligned}$$

and the initial conditions $\Delta u_{ni}(0) = 0$, $i = 1,2,\ldots,n$.

Here for $i = 1,2,\ldots,n$ we use the notation

$$\psi_i(t) = \Bigg[a\bigg(\frac{2l}{\pi}\sum_{j=1}^{\infty}\frac{1}{2j-1}\,u_{2j-1}(t)\bigg) - a\bigg(\frac{2l}{\pi}\sum_{j=1}^{n'}\frac{1}{2j-1}\,u_{2j-1}(t)\bigg)\Bigg]\bigg(\frac{i\pi}{l}\bigg)^2 u_i(t).$$

5 Auxiliary statements. Let us present statements about the fulfillment of some inequalities for the principal part of the exact solution. Let $\|\cdot\| = \|\cdot\|_{L_2(0,l)}$.

Lemma 1. *The inequalities*

$$\begin{aligned}
\Big\|\frac{\partial^k}{\partial x^k}\,p_n u(x,t)\Big\|^2 &\le c_{kn}(t), \quad k=1,2, \quad 0<t<T, \\
\big\|\big(u(x,t) - p_n u(x,t)\big)\big\|^2 &\le \sigma_n(t), \quad 0<t<T,
\end{aligned}$$

are valid, where

$$c_{kn}(t) = \Big\|\frac{d^k}{dx^k}\,p_n\varphi(x)\Big\|^2 + \frac{1}{2a_0}\int_0^t \Big\|\frac{\partial^{k-1}}{\partial x^{k-1}}\,p_n f(x,\tau)\Big\|^2 d\tau, \quad k=1,2,$$

$$\sigma_n(t) = \big\|\varphi(x,t) - p_n\varphi(x,t)\big\|^2 + \frac{1}{2a_0}\int_0^t \big\|f(x,\tau) - p_n f(x,\tau)\big\|^2 d\tau,$$

$$p_n\varphi(x) = \sum_{i=1}^{n}\varphi_i \sin\frac{i\pi x}{l}\,, \quad p_n f(x,t) = \sum_{i=1}^{n} f_i(t)\sin\frac{i\pi x}{l}\,.$$

Lemma 2. *The inequality*

$$\left\| \frac{\partial^2}{\partial x^2} u_n(x,t) \right\|^2 \le c_{2n}(t), \quad 0 < t < T,$$

is valid.

6 Main result.

We introduce into consideration the value

$$c_k = \left\| \frac{d^k}{dx^k} \varphi(x) \right\|^2 + \frac{1}{2a_0} \int\limits_0^T \left\| \frac{\partial^{k-1}}{\partial x^{k-1}} f(x,t) \right\|^2 dt, \quad k = 1, 2,$$

and formulate the result on the method accuracy.

Theorem. For the method error the following estimate

$$\|\Delta u_n(x,t)\|^2 \le \frac{lTL^2}{3a_0} \left(\frac{l}{\pi} \right)^4 c_1 \frac{\sigma_n(T)}{n^3} \exp\left(l^2 L \sqrt{\frac{2}{3} c_2} \right)$$

is true.

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