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ON ONE SYSTEM OF FOURTH-ORDER NONLINEAR INTEGRO-DIFFERENTIAL PARABOLIC EQUATION

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Abstract. The uniqueness and stability of the solution of the initial-boundary value problem for one system of fourth-order nonlinear parabolic integro-differential equations are investigated.

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In this paper, we consider the system of nonlinear parabolic integro-differential equations. The types of the system of nonlinear equations discussed in this article are partially derived from the description of real diffusion processes (see, for instance, [5]-[7] and references therein). The models of integro-differential type discussed in the presented work were first proposed in [3]. In particular, the corresponding fourth-order integro-differential equation is investigated [4].

In the rectangle $Q_T = [0,1] \times [0,T]$, where T is a positive constant, consider the following initial-boundary problem:

$$\frac{\partial u}{\partial t} + \frac{\partial^2}{\partial x^2}\left\{\left[1+\int_0^t \left(\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \left(\frac{\partial^2 v}{\partial x^2}\right)^2\right) d\tau\right] \frac{\partial^2 u}{\partial x^2}\right\} = f(x,t), \tag{1}$$

$$\frac{\partial v}{\partial t} + \frac{\partial^2}{\partial x^2}\left\{\left[1+\int_0^t \left(\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \left(\frac{\partial^2 v}{\partial x^2}\right)^2\right) d\tau\right] \frac{\partial^2 v}{\partial x^2}\right\} = f(g,t), \tag{2}$$

$$u(0,t) = u(1,t) = 0, \quad \frac{\partial^2 u}{\partial x^2}(0,t) = \frac{\partial^2 u}{\partial x^2}(1,t) = 0, \quad u(x,0) = u_0(x), \tag{3}$$

$$v(0,t) = v(1,t) = 0, \quad \frac{\partial^2 v}{\partial x^2}(0,t) = \frac{\partial^2 v}{\partial x^2}(1,t) = 0, \quad v(x,0) = v_0(x), \tag{4}$$

where f, g, u_0 and v_0 are the given functions of their arguments.

Let us show the solution to the problem (1)-(4) is stable with respect to the right-hand sides of $f(x,t)$ and $g(x,t)$ and the initial conditions $u_0(x), v_0(x)$. Multiply (1) by the function u and integrate on $[0,1]$, using twice the formula of the integration by parts, in the second term on the left, and taking into consideration to the boundary conditions (3), we get

$$\frac{d}{dt}\|u\|^2 + \int_0^1 \left\{\left[1+\int_0^t \left(\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \left(\frac{\partial^2 v}{\partial x^2}\right)^2\right) d\tau\right] \frac{\partial^2 u}{\partial x^2}\right\} \frac{\partial^2 u}{\partial x^2} dx = \int_0^1 f(x,t) u dx.$$

For the right-hand side we using relation $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$ and the Poincare-Friedrichs inequality [1], while for the left-hand side we neglect the positive member. After simple transformation, we receive:

$$\frac{d}{dt}\|u\|^2 + \int_0^t \left(\frac{\partial^2 u}{\partial x^2}\right)^2 dx \leq \frac{1}{2}\int_0^1 f^2 dx + \int_0^t \left(\frac{\partial^2 u}{\partial x^2}\right)^2 dx,$$

or

$$\frac{d}{dt}\|u\|^2 + \left\|\frac{\partial^2 u}{\partial x^2}\right\|^2 \leq \|f\|^2.$$

After integrating with respect to variable t, taking into account the initial condition [3] and neglecting the positive term, in the above inequality, we have

$$\|u(t)\|^2 \leq \int_0^1 \|f(\tau)\|^2 d\tau + \|u_0\|^2.$$

By similar reasoning, we get an estimate for $v(x,t)$:

$$\|v(t)\|^2 \leq \int_0^1 \|g(\tau)\|^2 d\tau + \|v_0\|^2.$$

The last two estimations prove the stability of the solution of problem (1)-(4).

Now, show that if the initial-boundary value problem (1)-(4) has a solution, it is unique. Suppose, u_1 and u_2 are two solution of the first equation, and v_1 and v_2 are two solution of the second equation, for $w = u_1 - u_2$ and $z = v_1 - v_2$, we have:

$$\begin{aligned} &\frac{\partial w}{\partial t} + \frac{\partial^2}{\partial x^2}\left\{\left[1 + \int_0^t \left(\left(\frac{\partial^2 u_1}{\partial x^2}\right)^2 + \left(\frac{\partial^2 v_1}{\partial x^2}\right)^2\right) d\tau\right] \frac{\partial^2 u_1}{\partial x^2}\right. \\ &\left. - \left[1 + \int_0^t \left(\left(\frac{\partial^2 u_2}{\partial x^2}\right)^2 + \left(\frac{\partial^2 v_2}{\partial x^2}\right)^2\right) d\tau\right] \frac{\partial^2 u_2}{\partial x^2}\right\} = 0, \end{aligned} \tag{5}$$

$$\begin{aligned} &\frac{\partial z}{\partial t} + \frac{\partial^2}{\partial x^2}\left\{\left[1 + \int_0^t \left(\left(\frac{\partial^2 u_1}{\partial x^2}\right)^2 + \left(\frac{\partial^2 v_1}{\partial x^2}\right)^2\right) d\tau\right] \frac{\partial^2 v_1}{\partial x^2}\right. \\ &\left. - \left[1 + \int_0^t \left(\left(\frac{\partial^2 u_2}{\partial x^2}\right)^2 + \left(\frac{\partial^2 v_2}{\partial x^2}\right)^2\right) d\tau\right] \frac{\partial^2 v_2}{\partial x^2}\right\} = 0, \end{aligned} \tag{6}$$

$$w(0,t) = w(1,t) = 0, \quad \frac{\partial^2 w}{\partial x^2}(0,t) = \frac{\partial^2 w}{\partial x^2}(1,t) = 0, \quad w(x,0) = 0, \tag{7}$$

$$z(0,t) = z(1,t) = 0, \quad \frac{\partial^2 z}{\partial x^2}(0,t) = \frac{\partial^2 z}{\partial x^2}(1,t) = 0, \quad z(x,0) = 0, \tag{8}$$

Multiply equation (5) by the function w and integrate the obtained equation by $[0, 1]$. Using the formula of integration by parts twice for the second term on the left-hand side of the equation and taking into consideration the boundary conditions (7), we get

$$\frac{1}{2}\frac{d}{dt}\|w\|^2 + \int_0^1 \left(\frac{\partial^2 u}{\partial x^2}\right)^2 \left[\int_0^t \left(\left(\frac{\partial^2 u_1}{\partial x^2}\right)^2 + \left(\frac{\partial^2 v_1}{\partial x^2}\right)^2\right) d\tau \frac{\partial^2 u_1}{\partial x^2}\right.$$

$$\left. - \int_0^t \left(\left(\frac{\partial^2 u_2}{\partial x^2}\right)^2 + \left(\frac{\partial^2 v_2}{\partial x^2}\right)^2\right) d\tau \frac{\partial^2 u_2}{\partial x^2}\right] \left[\frac{\partial^2 u_1}{\partial x^2} - \frac{\partial^2 u_2}{\partial x^2}\right] = 0.$$

Let us apply to above equation the relation $(ca - db)(a - b) \geq (1/2)(c - d)(a^2 - b^2)$, assuming that

$$a = \frac{\partial^2 u_1}{\partial x^2}, \quad b = \frac{\partial^2 u_2}{\partial x^2}.$$

$$c = \int_0^t \left(\left(\frac{\partial^2 u_1}{\partial x^2}\right)^2 + \left(\frac{\partial^2 v_1}{\partial x^2}\right)^2\right) d\tau, \quad d = \int_0^t \left(\left(\frac{\partial^2 u_2}{\partial x^2}\right)^2 + \left(\frac{\partial^2 v_2}{\partial x^2}\right)^2\right) d\tau,$$

If we neglect the non-negative member, we get

$$\frac{1}{2}\frac{d}{dt}\|w\|^2 + \frac{1}{2}\int_0^1 \int_0^t \left[\left(\frac{\partial^2 u_1}{\partial x^2}\right)^2 - \left(\frac{\partial^2 u_2}{\partial x^2}\right)^2 + \left(\frac{\partial^2 v_1}{\partial x^2}\right)^2 - \left(\frac{\partial^2 v_2}{\partial x^2}\right)^2\right] d\tau$$

$$\times \left[\left(\frac{\partial^2 u_1}{\partial x^2}\right)^2 - \left(\frac{\partial^2 u_2}{\partial x^2}\right)^2\right] dx \leq 0.$$

By similar reasoning, we get

$$\frac{1}{2}\frac{d}{dt}\|z\|^2 + \frac{1}{2}\int_0^1 \int_0^t \left[\left(\frac{\partial^2 u_1}{\partial x^2}\right)^2 - \left(\frac{\partial^2 u_2}{\partial x^2}\right)^2 + \left(\frac{\partial^2 v_1}{\partial x^2}\right)^2 - \left(\frac{\partial^2 v_2}{\partial x^2}\right)^2\right] d\tau$$

$$\times \left[\left(\frac{\partial^2 v_1}{\partial x^2}\right)^2 - \left(\frac{\partial^2 v_2}{\partial x^2}\right)^2\right] dx \leq 0.$$

Add the last two inequalities

$$\frac{1}{2}\frac{d}{dt}\|w\|^2 + \frac{1}{2}\frac{d}{dt}\|z\|^2$$

$$+\frac{1}{2}\int_0^1 \int_0^t \left[\left(\frac{\partial^2 u_1}{\partial x^2}\right)^2 - \left(\frac{\partial^2 u_2}{\partial x^2}\right)^2 + \left(\frac{\partial^2 v_1}{\partial x^2}\right)^2 - \left(\frac{\partial^2 v_2}{\partial x^2}\right)^2\right] d\tau$$

$$\times\left[\left(\frac{\partial^2 v_1}{\partial x^2}\right)^2-\left(\frac{\partial^2 v_2}{\partial x^2}\right)^2+\left(\frac{\partial^2 v_1}{\partial x^2}\right)^2-\left(\frac{\partial^2 v_2}{\partial x^2}\right)^2\right]dx\leq 0.$$

Using the following notation

$$\varphi(x,t)=\int_0^1\left[\left(\frac{\partial^2 u_1}{\partial x^2}\right)^2-\left(\frac{\partial^2 u_2}{\partial x^2}\right)^2+\left(\frac{\partial^2 v_1}{\partial x^2}\right)^2-\left(\frac{\partial^2 v_2}{\partial x^2}\right)^2\right]d\tau,$$

finally we arrive at

$$\frac{1}{2}\frac{d}{dt}\|z\|^2+\frac{1}{2}\frac{d}{dt}\|w\|^2+\frac{1}{4}\frac{d}{dt}\int_0^1\varphi^2dx\leq 0.$$

After integrating with respect to t and taking into account the initial conditions (7) and (8) the third term in this inequality gives us a non-negative member, and if we ignore it, we get $\|w\|^2+\|z\|^2\leq 0$. Thus, $w=z\equiv 0$, which proves the uniqueness of the solution of problem (1)-(4).

REFERENCES

1. Friedman, A. Partial Differential Equations of Parabolic Type. *Prentice-Hall,* 1964.
2. Dzhangveladze, T.A. First boundary-value problem for a nonlinear equation of parabolic type (Russian). *Dokl. Akad. Nauk SSSR,* **269**, 4 (1983), 839-842. *English translation: Soviet Phys. Dokl.,* **28**, 4 (1983), 323-324.
3. Gordeziani, D., Dzhangveladze, T., Korshia, T. Existence and uniqueness of the solution of a class of nonlinear parabolic problems (Russian). Differ. *Uravn,* **19**, 7 (1983), 1197-1207. *English translation: Differ. Equ.,* **19**, 7 (1984), 887-895.
4. Jangveladze, T. Investigation and numerical solution of nonlinear partial differential and integro-differential models based on system of Maxwell equations. *Semin. I.Vekua Inst. Appl. Math. Rep.,* **23** (1997), 51-87.
5. Jangveladze, T., Kiguradze, Z., Neta, B. Numerical Solution of Three Classes of Nonlinear Parabolic Integro-Differential Equations. *Mem. Differential Equations Math. Phys.*, **76** (2019), 1-118.
6. Jangveladze, T., Kiguradze, Z., Neta, B. Numerical Solution of Three Classes of Nonlinear Parabolic Integro-Differential Equations, Elsevier, 2016, ACADEMIC PRESS, ISBN: 978-0-12-804628-9. Elsevier/Academic Press, Amsterdam, 2015.
7. Chkhikvadze, T. On one nonlinear integro-differential parabolic equation. *Rep. Enlarged Sess. Semin. I. Vekua Inst. Appl. Math.,* **35** (2021), 19-22.

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