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THE BENDING CONTACT PROBLEM OF ELASTIC RECTANGULAR PLATE WITH RIGID INCLUSION

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Abstract. In the article bending problem of elastic rectangular plate with rigid inclusion is considered. The problem is formulated in the form of the integral equation, with respect to the jump of lateral force. Using the method of orthogonal polynomial, the integral equation is reduced to the infinite system of linear algebraic equations. Quasi-regularity of this system is proved.

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1 Introduction. The contact problems of bending of finite or infinite isotropic plate, reinforced with rigid or elastic inclusion, were solved by many authors, for example [1–3] . The bending problem of elastic rectangular plate with rigid inclusion is considered. The middle surface of rectangular elastic plate occupies the area $\Omega\backslash I$, where $\Omega = \left\{(x,y)|\,|x|<\frac{a}{2}, 0<y<b\right\}$ and $I=\left\{(x,y)\,|\,|x|<c, y=\frac{b}{2}, c<\frac{a}{2}\right\}$. Along the interval I the plate is reinforced with rigid inclusion, which is loaded by normal stress, with intensity $\mu_0\left(x\right)$. The bending function $\omega\left(x,y\right)$ satisfies the homogeneous biharmonic equation:

$$\Delta^2\omega\left(x,y\right)=0,\ in\ \Omega\backslash I. \tag{1}$$

The boundary of the plate is simply supported and the boundary conditions have the form:

$$\omega=0, M_x=0\ at\ x=\pm\frac{a}{2};\quad \omega=0, M_y=0\ at\ y=0\ and\ y=b. \tag{2}$$

The jumps of bending function, angle of rotation, bending moment and lateral force on the inclusion are presented

$$<\omega>=<\omega'_y>=<M_y>=0\ ,\ <N_y>=\mu\left(x\right)\ \ at\ \ |x|<c, \tag{3}$$

where $<f>=f\left(x,\frac{b}{2}^-\right)-f\left(x,\frac{b}{2}^+\right)$ and $\mu\left(x\right)$ is an unknown function. The contact between the plate and the inclusion is realized by a thin glue layer. The contact condition has the form [4]:

$$\omega_0\left(x\right)-\omega\left(x,\frac{b}{2}\right)=n_0\mu\left(x\right)\ \ at\ \ |x|<c \tag{4}$$

and the bending function of rigid inclusion satisfies the following condition:

$$\omega_0\left(x\right)=c_0x+d\ at\ \ |x|<c, \tag{5}$$

where c_0 , d are known constants, n_0 is a glue parameter. The equilibrium equations of inclusion have the following form:

$$\int_{-c}^{c} (\mu(t) - \mu_0(t))\, dt = 0 \ ; \ \int_{-c}^{c} t\,(\mu(t) - \mu_0(t))\, dt = 0, \tag{6}$$

where $\mu_0(t)$ is known function, which express exterior load of inclusion.

2 Solution of the problem. We are finding the solution of the biharmonic equation (1) from the class of even functions, with respect to the variable x, in the form:

$$\omega(x, y) = \sum_{k=1,2,3}^{\infty} \cos(\alpha_k x) Y_k(y), \tag{7}$$

where

$$\alpha_k = \frac{\pi(2k-1)}{a};$$

$$Y_k(y) = \begin{cases} A_k sinh(\alpha_k y) + \alpha_k B_k y cosh(\alpha_k y) & 0 \le y < \frac{b}{2} \\ C_k sinh(\alpha_k (b-y)) + \alpha_k D_k (b-y)\, cosh(\alpha_k (b-y)) & \frac{b}{2} < y \le b \end{cases} \tag{8}$$

A_k, B_k, C_k, D_k are unknown constants. Implement the jump condition (3), the relations (7) and (8) give that $A_k = C_k$, $B_k = D_k$ and

$$A_k = \frac{cosh(\frac{\alpha_k b}{2}) + \alpha_k b sinh(\frac{\alpha_k b}{2})}{2\alpha_k{}^3 cosh^2(\frac{\alpha_k b}{2})} \frac{1}{2aD} \int_{-c}^{c} \mu(x)\, cos(\alpha_k x) dx \ ;$$

$$B_k = \frac{1}{2\alpha_k{}^3 cosh^2(\frac{\alpha_k b}{2})} \frac{1}{2aD} \int_{-c}^{c} \mu(x)\, cos(\alpha_k x) dx,$$

where D is a cylindrical rigidity of the plate. Using obtained results $\omega\left(x, \frac{b}{2}\right)$ is rewritten as follows

$$\omega\left(x, \frac{b}{2}\right) = \frac{1}{2aD} \sum_{k=1,2,3}^{\infty} \int_{-c}^{c} \frac{\cos \alpha_k (x-\xi)}{\alpha_k{}^3} \rho_k \mu(\xi)\, d\xi, \tag{9}$$

where

$$\rho_k = tanh \frac{\alpha_k b}{2} - \frac{\alpha_k b}{2cosh^2 \frac{\alpha_k b}{2}}. \tag{10}$$

The relation (9) is represented as the sum of the main and regular parts

$$\omega\left(x, \frac{b}{2}\right) = \frac{1}{2\pi D} \int_{-c}^{c} \frac{(x-\xi)^2}{4} \ln \frac{1}{|x-\xi|} \mu(\xi)\, d\xi + \int_{-c}^{c} R(x, \xi)\, \mu(\xi)\, d\xi, \tag{11}$$

where $R(x,\xi) = \frac{1}{2aD}B_0(x-\xi) + \frac{1}{2aD}\sum\limits_{k=1,3}^{\infty}\frac{\cos\alpha_k(x-\xi)}{\alpha_k{}^3}(\rho_k-1)$;

$B_0(t) = -\frac{t^2}{4}(3+\ln 2) + \sum\limits_{k=1,3}^{\infty}\frac{1}{k^3} + \sum\limits_{n=1,3}^{\infty}\frac{2^{2n-1}\tilde{B}_n t^{2n+2}}{(2n+2)!2n}$ and $\tilde{B}_n$ are Bernoulli numbers.

Using conditions (4), (5) the following integral equation with respect to the unknown function $\nu(s)$ is obtained

$$-\frac{c^3}{8\pi D}\int\limits_{-1}^{1}(s-\eta)^2\ln c\,|s-\eta|\nu(\eta)\,d\eta - \int\limits_{-1}^{1}R_0(s,\eta)\,\nu(\eta)\,cd\eta = n_0\nu_0(s) - \tilde{c}s - d; \quad (12)$$

$$\int\limits_{-1}^{1}[\nu(s)-\nu_0(s)]\,ds = 0\ ,\ \int\limits_{-1}^{1}s\,[\nu(s)-\nu_0(s)]\,ds = 0, \quad (13)$$

where $x = cs$, $\xi = c\eta$, $\nu(\eta) = \mu(c\eta)$, $\nu_0(\eta) = \mu_0(c\eta)$,$R_0(s,\eta) = cR(cx,\xi)$, $\tilde{c} = c_0c$.

Based of the symmetry of the problem with respect to x , the constant $\tilde{c}$ equals zero and the solution of the problem $\nu(\eta)$ is sought in the class of even function. Accordingly, the second condition of (13) is fulfilled automatically. Thus, problem (12), (13) take the form:

$$-\frac{c^3}{8\pi D}\int\limits_{-1}^{1}(s-\eta)^2\ln c\,|s-\eta|\nu(\eta)\,d\eta - \int\limits_{-1}^{1}R_0(s,\eta)\,\nu(\eta)\,d\eta = n_0\nu(s) - d; \quad (14)$$

$$\int\limits_{-1}^{1}\nu(s)\,ds = P_0, \quad (15)$$

where $P_0 \equiv \int\limits_{-1}^{1}\nu_0(s)\,ds$.

Let us present the solution of the problem (14), (15) as follows

$$\nu(s) = A + \sqrt{1-s^2}\sum_{k=1}^{\infty}X_{2k}P_{2k}^{\left(\frac{1}{2},\frac{1}{2}\right)}(s), \quad (16)$$

where $P_{2k}^{\left(\frac{1}{2},\frac{1}{2}\right)}(s)$ are Chebyshev's polynomials of the second kind and X_{2k} , $k = 1,2,3,...$ are unknown constants, from condition (15) we have $A = \frac{P_0}{2}$. Using Rodrigue's and Tricomi formulas for Jacobi's polynomials, based on the orthogonality condition of Chebyshev's [5] polynomials we obtain the following infinite system of linear algebraic equations

$$n_0Y_{2n} - \sum_{k=1}^{\infty}\left[\lambda_1R_{nk}^{(1)} + \lambda_2R_{nk}^{(2)}\right]\frac{Y_{2k}}{h_{2k}k\,(2k-1)\,(2k-2)} = f_{2n}, n = 1,2,3,... \quad (17)$$

where

$$R_{nk}^{(1)} = \int\limits_{-1}^{1} P_{2k+4}^{\left(-\frac{7}{2},-\frac{7}{2}\right)}(y)\, P_{2n}^{\left(\frac{1}{2},\frac{1}{2}\right)}(y)\, dy \ ; \ R_{nk}^{(2)} = \int\limits_{-1}^{1} \tilde{R}_{2k}(y)\, P_{2n}^{\left(\frac{1}{2},\frac{1}{2}\right)}(y)\, dy \ , \ h_{2n}X_{2n} \equiv Y_{2n} \ ;$$

$$\tilde{R}_{2k}(y) = \int\limits_{-1}^{1} \frac{\partial^3 R_0(y,\eta)}{\partial \eta^3} P_{2k+4}^{\left(-\frac{7}{2},-\frac{7}{2}\right)}(\eta)\, d\eta \ ; \ f_{2n} \equiv -\int\limits_{-1}^{1} \left(f(y) + \frac{P_0 n_0}{2} + d \right) P_{2n}^{\left(\frac{1}{2},\frac{1}{2}\right)}(y)\, dy \ ;$$

$$h_{2n} \equiv \frac{2}{2n+1} \frac{\Gamma^2(2n+\frac{3}{2})}{\Gamma(2n+1)\Gamma(2n+2)},$$

$P_n^{(\alpha,\beta)}(s)$ is Jacobi orthogonal polynomials, $\Gamma(z)$ is a known gamma function. Let's investigate the system (17) for quasi-regularity.

Using Darboux's asymptotic formula for Jacobi's polynomials and Stirling's formula for gamma function we obtain the following estimates

$$S_n = \sum_{k=1}^{\infty} \frac{\left|\lambda_1 R_{nk}^{(1)} + \lambda_2 R_{nk}^{(2)}\right|}{h_{2k} k\,(2k-1)\,(2k-2)} \to 0, \ when\ n \to \infty\ and\ f_{2n} \to 0, \ when\ n \to \infty \qquad (18)$$

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3 Conclusions.

The estimates (18) show that the infinite system of linear algebraic equations (17) is quasi-regularity. The reduction method is justified and it is possible to find an approximate solution with any accuracy [6].

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