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ON ONE NONLINEAR DIFFUSION SYSTEM

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Abstract. The asymptotic behavior, as time variable tends to infinity, of a solution for a nonlinear diffusion system is considered. It is shown that the stationary solution of the system is linearly stable, and the possibility of the Hopf-type bifurcation is observed.

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Most scientific problems and phenomena such as diffusion, heat transfer, fluid mechanics, plasma physics, plasma waves, thermo-elasticity and chemical physics occur nonlinearly. The Diffusion Equation is a partial differential equation that describes the density fluctuations in a diffusing material.

In this paper we study the behavior of a solution for a one-dimensional nonlinear diffusion system. The existence of Hopf bifurcation to partial differential equation models (see, for example, [5]) are derived, also. The linear stability and Hopf bifurcation of a solution of the initial-boundary value problem investigated in this article models at first appeared in [2]. Similar issues have been studied on various widespread models in the works [3], [4], [6] (for extensive citation and annotation, see, for example [7]).

The system of equations discussed in this article is a generalized version of the system given in [1].

For the given system of equations, in the domain $Q_t = (0,1) \times (0,t)$, consider the following initial-boundary value problem:

$$\begin{gathered}\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left(S^{\alpha}\frac{\partial U}{\partial x}\right), \quad \frac{\partial V}{\partial t} = \frac{\partial}{\partial x}\left(S^{\alpha}\frac{\partial V}{\partial x}\right), \\ \frac{\partial S}{\partial t} = -aS^{\beta} + bS^{\gamma}\left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] + cS^{\gamma}\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial x}\right),\end{gathered} \tag{1}$$

$$\begin{gathered}U(0,t) = V(0,t) = 0, \\ U(1,t) = \psi_1 > 0, \quad V(1,t) = \psi_2 > 0,\end{gathered} \tag{2}$$

$$U(x,0) = U_0(x), \quad V(x,0) = V_0(x), \quad S(x,0) = S_0(x) > 0, \tag{3}$$

where α, β, γ are real numbers, a, b, c, ψ_1, ψ_2 are positive real numbers, and $U_0(x)$, $V_0(x)$, $S_0(x)$ are given functions.

When $\beta \neq \gamma$, the unique stationary solution $(U_s, V_s,\ S_s)$ of problem (1)-(3) is:

$$U_s = \psi_1 x, \quad V_s = \psi_2 x, \quad S_s = \left[\frac{b}{a}\left({\psi_1}^2 + {\psi_2}^2\right) + \frac{c}{a}\left(\psi_1 + \psi_2\right)\right]^{\frac{1}{\beta-\gamma}}. \tag{4}$$

Let us examine if the stationary solution (4) of the problem (1)-(3) is linearly stable. Rewrite the solutions of problem (1)-(3) in the following form:

$$\begin{aligned} U(x,t) &= U_s + u(x,t), \\ V(x,t) &= V_s + v(x,t), \\ S(x,t) &= S_s + s(x,t), \end{aligned} \tag{5}$$

where $u(x,t)$, $v(x,t)$, $s(x,t)$ are small perturbations. The system (1) takes the following form:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \alpha_s \frac{\partial s}{\partial x} + \beta_s \frac{\partial^2 u}{\partial x^2}, \\ \frac{\partial v}{\partial t} &= \gamma_s \frac{\partial s}{\partial x} + \beta_s \frac{\partial^2 v}{\partial x^2}, \\ \frac{\partial s}{\partial t} &= \upsilon_s s + \eta_s \frac{\partial u}{\partial x} + \mu_s \frac{\partial v}{\partial x}, \end{aligned}$$

where:

$$\alpha_s = \psi_1 \alpha \left[\frac{b}{a}\left({\psi_1}^2 + {\psi_2}^2\right) + \frac{c}{a}\left(\psi_1 + \psi_2\right)\right]^{\frac{\alpha-1}{\beta-\gamma}}, \gamma_s = \psi_2 \alpha \left[\frac{b}{a}\left({\psi_1}^2 + {\psi_2}^2\right) + \frac{c}{a}\left(\psi_1 + \psi_2\right)\right]^{\frac{\alpha-1}{\beta-\gamma}},$$

$$\beta_s = \left[\frac{b}{a}\left({\psi_1}^2 + {\psi_2}^2\right) + \frac{c}{a}\left(\psi_1 + \psi_2\right)\right]^{\frac{\alpha}{\beta-\gamma}}, \upsilon_s = (\gamma - \beta)\, a \left[\frac{b}{a}\left({\psi_1}^2 + {\psi_2}^2\right) + \frac{c}{a}\left(\psi_1 + \psi_2\right)\right]^{\frac{\beta-1}{\beta-\gamma}},$$

$$\eta_s = (2b\psi_1 + c) \left[\frac{b}{a}\left({\psi_1}^2 + {\psi_2}^2\right) + \frac{c}{a}\left(\psi_1 + \psi_2\right)\right]^{\frac{\gamma}{\beta-\gamma}},$$

$$\mu_s = (2b\psi_2 + c) \left[\frac{b}{a}\left({\psi_1}^2 + {\psi_2}^2\right) + \frac{c}{a}\left(\psi_1 + \psi_2\right)\right]^{\frac{\gamma}{\beta-\gamma}}.$$

Introduce the following notations:

$$u(x,t) = \bar{u}(x)\, e^{\omega t}, \quad v(x,t) = \bar{v}(x)\, e^{\omega t}, \quad s(x,t) = \bar{s}(x)\, e^{\omega t}, \tag{6}$$

where $\bar{u}(x) = u_0 e^{ikx}$, $\bar{v}(x) = v_0 e^{ikx}$, $\bar{s}(x) = s_0 e^{ikx}$. After appropriate transformations, we get:

$$\begin{aligned} \left(\omega + \beta_s k^2\right) u_0 - \alpha_s i k s_0 &= 0, \\ \left(\omega + \beta_s k^2\right) v_0 - \gamma_s i k s_0 &= 0, \\ \eta_s i k u_0 + \mu_s i k v_0 + (\upsilon_s - \omega)\, s_0 &= 0, \end{aligned}$$

which has a nontrivial solution, if its main determinant $\Delta(\omega, k) = 0$. So,

$$\left(\omega + \beta_s k^2\right)\left[\left(\omega + \beta_s k^2\right)(v_s - \omega) - k^2\eta_s\alpha_s - k^2\mu_s\gamma_s\right] = 0.$$

Consider the following equality:

$$k^2\left(\beta_s v_s - \beta_s\omega - \alpha_s\eta_s - \gamma_s\mu_s\right) - \omega^2 + v_s\omega = 0. \tag{7}$$

Solving the equation (7) with respect to k gives $k_1 = -k_2$. Subsequently, we get

$$\begin{aligned} \bar{u}(x) &= \frac{ik_1\alpha_s}{\omega + \beta_s k_1^2}\left(S_1 e^{ik_1x} - S_2 e^{-ik_1x}\right), \\ \bar{v}(x) &= \frac{ik_1\gamma_s}{\omega + \beta_s k_1^2}\left(S_1 e^{ik_1x} - S_2 e^{-ik_1x}\right), \\ \bar{s}(x) &= S_1 e^{ik_1x} + S_2 e^{-ik_1x}. \end{aligned} \tag{8}$$

Taking into consideration the boundary conditions (2) and equalities (5), (6), we have

$$\bar{u}(0) = \bar{u}(1) = 0.$$

From this and (8), it follows that

$$S_1 - S_2 = 0,$$
$$S_1 e^{ik_1x} - S_2 e^{-ik_1x} = 0.$$

This system has a nonzero solution, when

$$\Delta(\omega, k) = \begin{vmatrix} 1 & -1 \\ e^{ik_1} & -e^{-ik_1} \end{vmatrix} = 2isink_1 = 0,$$

from where it follows that $k_{1n} = \pi n,\ n\epsilon\ Z$.

Rewrite equality (7) in the following form:

$$\omega_n^2 + P_n\left(\beta_s, k_n, v_s\right)\omega_n + L_n\left(\alpha_s, \beta_s, k_n, v_s, \eta_s, \mu_s, \gamma_s\right) = 0,$$

where:

$$P_n\left(\beta_s, k_n, v_s\right) = \beta_s k_n^2 - \ v_s,$$
$$L_n\left(\alpha_s, \beta_s, k_n, v_s, \eta_s, \mu_s, \gamma_s\right) = -\beta_s v_s k_n^2 + \alpha_s\eta_s k_n^2 + \gamma_s\mu_s k_n^2.$$

If the condition $Re(\omega_n) < 0$ holds for each n, then the stacionary solution of the problem (1)(3) is linearly stable. When $2\alpha + \beta - \gamma > 0$, then $L_n > 0$, i.e. $P_n > 0$.

Therefore, the following statement is true.

Theorem. *If $2\alpha + \beta - \gamma > 0, \beta \neq \gamma$, then stationary solution (4) of problem (1)-(3) to be linearly stable, it is necessary and sufficient that the following inequality holds*

$$a(\gamma - \beta)\left[\frac{b}{a}\left({\psi_1}^2 + {\psi_2}^2\right) + \frac{c}{a}\left(\psi_1 + \psi_2\right)\right]^{\frac{\beta-\alpha-1}{\beta-\gamma}} < \pi^2.$$

Remark. From the last inequality, it is evident, that when $\gamma < \beta$, the solution of problem (1)-(3) is always linearly stable.

Suppose, $\gamma > \beta$, $\beta - \alpha - 1 \neq 0$ and $\psi_1 = \psi_2 = \psi$. Then

$$\psi_c = \frac{-c + \sqrt{c^2 + 2ab\left(\frac{\pi^2}{a(\gamma-\beta)}\right)^{\frac{\beta-\gamma}{\beta-\alpha-1}}}}{2b}.$$

For which the following relations are true:

$$P_1\left(\psi_c, \alpha, \beta, \gamma\right) = 0, \quad P_n\left(\psi_c, \alpha, \beta, \gamma\right) > 0, \quad n = 2,\ 3,\ \ldots\ .$$

In addition, if we assume, that $\beta - \alpha - 1 < 0$, then for $\psi \in (0, \psi_c), \psi = \psi_1 = \psi_2$ we have $P_n\left(\psi, \alpha, \beta, \gamma\right) > 0$, $n \in Z_0$.

Thus, if $\psi \in (0, \psi_c)$, then the stacionary solution of problem (1)-(3) is always linearly stable; while if $\psi \in (\psi_c, +\infty)$, then is nonstable. If $\psi = \psi_c$, then $Re(\omega_1) = 0$ and $Im(\omega_1) \neq 0$ which means that the possibility of Hoph bifurcation occurs; That is the small perturbations may cause transformation of a solution in periodic oscillations.

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