Reports of Enlarged Sessions of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 36, 2022

ON THE GENERALIZED ABSOLUTE CONVERGENCE OF DOUBLE FOURIER–HAAR SERIES

Rusudan Meskhia

Abstract. The sufficient conditions for the generalized absolute convergence of double Fourier–Haar series are established in terms of mixed and partial moduli of δ-variation of the function of two variables.

Keywords and phrases: Double Fourier–Haar series, the modulus of δ-variation.

AMS subject classification (2010): 42A20, 26A16.

1 Introduction The questions dealing with the absolute convergence of Fourier-Haar series have been investigated in the works B. Golubov [4], Z. Chanturia [1] and many other authors.

2 Content The problem of convergence of the series

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\gamma_{mn}|\widehat{f}(m,n)|^{\Gamma}, \quad 0<r<2,$$

is considered, where $\{\gamma_{mn}\}_{m\geq 1,\, n\geq 1}$ is a definite multiple sequence of nonnegative numbers and

$$\widehat{f}(m,n)=\iint\limits_{I^2} f(x,y)\lambda_m(x)\lambda_n(y)\,dx\,dy$$

are the Fourier–Haar coefficients of the function $f(x,y)\in L(I^2)$, where $I^2=[0,1]\times[0,1]$ and $\lambda_m(x)\lambda_n(y)\}_{m\geq 1,\, n\geq 1}$ is the multiple Haar system [4].

Following the definition in [3] and notations of [8] a sequence $\{\gamma_{kj}\}_{k\geq 1,\, j\geq 1}$, $k,j\in\mathbb{N}$ of nonnegative numbers is said to belong to the class A_α, for some $\alpha>1$, if

$$\Big(\sum_{k\in D_m}\sum_{j\in D_n}\gamma_{kj}^{\alpha}\Big)^{\frac{1}{\alpha}}\leq c\cdot 2^{\frac{(m+n)(1-\alpha)}{\alpha}}\sum_{k\in D_{m-1}}\sum_{j\in D_{n-1}}\gamma_{kj},$$

$$\Big(\sum_{k\in D_m}\gamma_{k1}^{\alpha}\Big)^{\frac{1}{\alpha}}\leq c_1\cdot 2^{\frac{m(1-\alpha)}{\alpha}}\sum_{k\in D_{m-1}}\gamma_{k1},$$

$$\Big(\sum_{j\in D_n}\gamma_{1j}^{\alpha}\Big)^{\frac{1}{\alpha}}\leq c_2\cdot 2^{\frac{n(1-\alpha)}{\alpha}}\sum_{j\in D_{n-1}}\gamma_{1j},$$

where $D_0 = D_1 = \{1\}$, $D_i = \{2^{i-1}+1, 2^{i-1}+2, \dots, 2^i\}$, $i \in \mathbb{N}$, and the constants c, c_1, c_2 depend only on α.

$B(I^2)$ denotes class of bounded functions on the I^2.

$BV_s(I^2)$, $s \geq 1$ is the class of the functions with bounded s variation on the I^2 [4].

$\upsilon(m,n;f)$-denotes the mixed modulus of variation of the function $f \in B(I^2)$.

$\upsilon_1(m;f)$, $\upsilon_2(n;f)$ are partial moduli of variation of $f \in B(I^2)$.

The definitions of mixed and partial moduli of variation for the function of two variables was introduced by Kraszkowski [6] according to Chanturia's [2] modulus of variation. $\varphi(m,n;\delta_1,\delta_2;f)$-denotes mixed modulus of $\delta(\delta_1,\delta_2)$ variation of the function $f \in B(I^2)$, $\varphi_1(m;\delta_1;f)$ and $\varphi_2(n;\delta_2;f)$ are partial moduli of δ-variation. The mixed and partial moduli of δ-variation of the function $f(x,y) \in B(I^2)$ are defined, according to Karchava's [5] modulus of δ-variation, in the following way:

$$\varphi(m,n;\delta_1,\delta_2;f) = \sup_{\Pi_{m,n;\delta_1,\delta_2}} \sum_{n=1}^{m} \sum_{j=1}^{n} \omega(f;I_k \times B_j),$$

$$\varphi_1(m;\delta_1;f) = \sup_{0\leq y\leq 1} \sup_{\Pi_{m;\delta_1}} \sum_{k=1}^{m} \omega_1(f;I_k),$$

$$\varphi_2(n;\delta_2;f) = \sup_{0\leq x\leq 1} \sup_{\Pi_{n;\delta_2}} \sum_{j=1}^{n} \omega_2(f;B_j),$$

where $m,n \in \mathbb{N}$, $\delta_1,\delta_2 > 0$,

$$\omega(f;I_k \times B_j) = \sup\Big\{\big|f(x+h_1,y+h_2) - f(x,y+h_2) - f(x+h_1,y) + f(x,y)\big| : (x,y),(x+h_1,y+h_2) \in I_k \times B_j,\ h_1,h_2 > 0\Big\},$$

$$\omega_1(f;I_k) = \sup\Big\{|f(x+h_1,y) - f(x,y)| :\ x, x+h_1 \in I_k,\ h_1 > 0\Big\},$$

$$\omega_2(f;B_j) = \sup\Big\{|f(x,y+h_2) - f(x,y)| :\ y, y+h_2 \in B_j,\ h_2 > 0\Big\},$$

$\Pi_{m,n;\delta_1,\delta_2}$ is an arbitrary system of mn pairwise nonintersecting rectangles $I_k \times B_j \subset I^2$, $1 \leq k \leq m$, $1 \leq j \leq n$, $k,j \in \mathbb{N}$.

$\Pi_{m;\delta_1}$ ($\Pi_{n;\delta_2}$) is an arbitrary system of nonintersecting intervals $\{I_k\}_{1\leq k\leq m}$ ($\{B_j\}_{1\leq j\leq n}$) of the segment $[0,1]$. The length of each interval I_k (B_j) is equal to δ_1 (δ_2).

The following statement is true.

Theorem. *Let* $\{\gamma_{kj}\} \in A_{\frac{p}{p-rp+r}}$ *for some numbers* $p > 1$ *and* $0 < r < 2$, $f(x,y) \in B(I^2)$ *and in addition*

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \gamma_{mn}(mn)^{-\frac{3}{2}r} \Bigg(\sum_{k=m+1}^{2m} \sum_{j=n+1}^{2n} \frac{\varphi(k,j;\frac{1}{2m},\frac{1}{2n};f)}{kj} \Bigg)^r < +\infty,$$

$$\sum_{m=1}^{\infty} \gamma_{m1} m^{-\frac{3}{2}r} \Bigg(\sum_{k=m+1}^{2m} \frac{\varphi_1(k;\frac{1}{2m};f)}{k} \Bigg)^r < +\infty,$$

$$\sum_{n=1}^{\infty} \gamma_{1n} n^{-\frac{3}{2}r} \Big(\sum_{j=n+1}^{2n} \frac{\varphi_2(j; \frac{1}{2n}; f)}{j} \Big)^r < +\infty,$$

then the series

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \gamma_{mn} |\widehat{f}(m,n)|^r$$

converges.

The Theorem presents the analogue of the theorem, obtained by Meskhia [7] for double Fourier–Haar series.

From the Theorem when $\gamma_{mn} = 1$, $m, n \in \mathbb{N}$ follows:

Corollary 1. *Let* $f(x,y) \in B(I^2)$ *and*

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (mn)^{-\frac{3}{2}r} \upsilon^r(m,n;f) < +\infty,$$

$$\sum_{m=1}^{\infty} m^{-\frac{3}{2}r} \upsilon_1^r(m;f) < +\infty,$$

$$\sum_{n=1}^{\infty} n^{-\frac{3}{2}r} \upsilon_2^r(n;f) < +\infty,$$

then

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |\widehat{f}(m,n)|^r < +\infty.$$

Corollary 1 is the analogue of Chanturia's [1] theorem for double Fourier–Haar series.

For $\gamma_{mn} = (mn)^\gamma$, $m, n \in \mathbb{N}$ the Theorem leads to Golubov's [4] theorem, which can be formulated as follows:

Corollary 2. *Let* $f(x,y) \in BV_s(I^2)$, $s \geq 1$, *then*

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (mn)^\gamma |\widehat{f}(m,n)|^r < +\infty,$$

when

$$\gamma + 1 < r\Big(\frac{1}{s} + \frac{1}{2}\Big), \quad \gamma \in \mathbb{R}, \quad 0 < r < 2.$$

REFERENCES

1. Chanturia, Z.A. On the absolute convergence of the series of Fourier–Haar coefficients. Special issue dedicated to Wladyslaw Orlicz on the occasion of his seventy-fifth birthday. *Comment. Math. Special Issue*, **2** (1979), 25–35.

2. Chanturia, Z.A. The modulus of variation of a function and continuity (Russian). *Sakharth. SSR Mecn. Akad. Moambe* **80**, 2 (1975), 281–283.

3. Gogoladze, L.D. Uniform strong summation of multiple trigonometric Fourier series (Russian). *Reports of the extended sessions of a seminar of the I. N. Vekua Institute of Applied Mathematics, Vol. I, no. 2 (Russian) (Tbilisi, 1985)*, 48–51, *Tbilis. Gos. Univ., Tbilisi*, 1985.

4. Golubov, B.I. Absolute convergence of double series of Fourier-Haar coefficients for functions of bounded p-variation. (Russian) *Izv. Vyssh. Uchebn. Zaved. Mat.*, **2012**, 6, 3–13; translation in *Russian Math. (Iz. VUZ),* **56**, 6 (2012), 1–10.

5. Karchava, T. On the absolute convergence of Fourier series. *Georgian Math. J.* **4**, 4 (1997), 333–340.

6. Kraczkowski, A. An application of double moduli of variation to double Fourier–Haar series. *Funct. Approx. Comment. Math.* **6** (1978), 119–133.

7. Meskhia, R. On the absolute convergence of Fourier–Haar series. *Proc. A. Razmadze Math. Inst.*, **157** (2011), 89–94.

8. Móricz, F., Veres, A. Absolute convergence of multiple Fourier series revisited. *Anal. Math.*, **34**, 2 (2008), 145–162.

Received 20.05.2022; accepted 23.08.2022.

Author(s) address(es):

Rusudan Meskhia
Faculty of Exact and Natural Sciences, I. Javakhishvili Tbilisi State University
University str. 2, Tbilisi 0186, Georgia
E-mail: rusudan.meskhia@tsu.ge