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LOCAL LIMIT THEOREM FOR SUMS OF DEPENDENT RANDOM VECTORS

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Abstract. On the probability space (Ω, F, P) a stationary in the narrow sense sequence $\{\xi_n, Y_n\}_{n\geq 1}$ is considered. $\{\xi_n\}_{n\geq 1}$ is a finite regular Markov chain.$\{Y_n\}_{n\geq 1}$is a sequence of random vectors with chain dependence. Local limit theorems are obtained for the conditional and unconditional distributions of the sums $S_{n_1} = \frac{1}{\sqrt{n}}\sum_{j=1}^{n}[Y_j - E(Y_j|\xi_j)]$ and $S_n = \frac{1}{\sqrt{n}}\sum_{j=1}^{n}[Y_j - E(Y_1)]$ respectively.

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1 Introduction. On a probability space (Ω, F, P) lets consider a two-component, stationary in a narrow sense, sequence

$$\{\xi_i, Y_i\}_{i\geq 1} \qquad (1)$$

$\{\xi_i\}_{i\geq 1}$ $(\xi_i : \Omega \longrightarrow \{1, 2, ..., s\})$, is a homogeneous, ergodic finite Markov chain. $P = \{p_{\alpha\beta}\}_{\alpha,\beta=\overline{1,s}}$ is its transition probability matrix and $\pi = (\pi_1, \pi_2, ..., \pi_s)$ is the initial distribution. $\{Y_i\}_{i\geq 1}$ $(Y_i : \Omega \longrightarrow R^k)$ is a sequence with chain dependence [1]. Lets the chain consist of one ergodic class (that may have cyclic subclasses).

Let's by the symbol $\xrightarrow{W}$ is designated a weak convergence. Let's introduce the notation:

$$\mu(\xi_j) = E(Y_j|\xi_j),\ R(\xi_j) = E\{[Y_j - \mu(\xi_j)][Y_j - \mu(\xi_j)]^T|\xi_j\},\ 1 \leq j \leq n;$$

$$\mu(\alpha) = E(Y_1|\xi_1 = \alpha),\ R(\alpha) = E\{[Y_1 - \mu(\alpha)][Y_1 - \mu(\alpha)]^T|\xi_1 = \alpha\},\ 1 \leq \alpha \leq s;$$

$$\mu = E\mu(\xi_1) = \sum_{\alpha=1}^{s}\pi_\alpha\mu(\alpha),\ R_0 = ER(\xi_1) = \sum_{\alpha=1}^{s}\pi_\alpha R(\alpha).$$

Let's consider sums

$$S_n = \frac{1}{\sqrt{n}}\sum_{j=1}^{n}[Y_j - \mu] = S_{n_1} + S_{n_2},\ S_{n_1} = \frac{1}{\sqrt{n}}\sum_{j=1}^{n}[Y_j - \mu(\xi_j)],\ S_{n_2} = \frac{1}{\sqrt{n}}\sum_{j=1}^{n}[\mu(\xi_j) - \mu] \quad (2)$$

Lemma 1. *(see [1]) Let's $sp(R_0) < \infty$ and for each function $\Psi : \Xi \rightarrow R^1$ for which $E\Psi(\xi_i) < \infty$, when $n \rightarrow \infty$ almost everywhere there is a convergence $\frac{1}{n}\sum_{j=1}^{n}\Psi(\xi_j) \rightarrow E\Psi(\xi_1)$ then the following limit equalities are true*

$$\mathrm{P}_{S_{n_1}} \xrightarrow{W} \Phi_{R_0}, \mathrm{P}_{S_{n_2}} \xrightarrow{W} \Phi_{T_\mu}, \mathrm{P}_{S_n} \xrightarrow{W} \Phi_{R_0+T_\mu}. \quad (3)$$

Here $T_\mu = F[\Pi_{dg}Z + (\Pi_{dg}Z)^T - \Pi_{dg}\Pi - \Pi_{dg}]_c F^T$. *Z is a fundamental matrix.*

$$F = \|\mu_{i\alpha}\|_{i=1,k,\alpha=1,s}, \mu_{i\alpha} = \mu_i(b_\alpha), \Pi = \|\pi_{\alpha\beta}\|_{\alpha,\beta=\bar{1,s}}, \pi_{\alpha,\beta} = \pi_\beta, \ \alpha,\beta = \overline{1,s}.$$

Let's $\nu_n(\alpha) = \nu_n(r_\alpha), (\alpha = \overline{1,s})$ *is a random variable that shows the number of time moments for the first n steps in that the chain is in the state* b_α*, when the trajectory* $\bar{\xi_{1n}} = (\xi_1, \xi_2, ..., \xi_n)$ *is fixed.*

Lemma 2. *(see [2]). For a regular chain, when* $\varepsilon > 0$ *are true the equalities*

$$\lim_{n\to\infty} E[\frac{\nu_n(\alpha)}{n}] = \pi_\alpha, E|\frac{\nu_n(\alpha)}{n} - \pi_\alpha|^2 \le \frac{c(\pi, P)}{n}, \lim_{n\to\infty} P\{|\frac{\nu_n(\alpha)}{n} - \pi_\alpha| > \varepsilon = 0\}, \ \ \alpha = \overline{1,s} \quad (4)$$

Let's introduce conditional distributions $P_\alpha \equiv P_{Y_1|\xi_1=\alpha}$, $\alpha = \overline{1,s}$ *of the quantity* ξ_1

2 Local limit theorem.

Theorem: When $sp(R_0) < \infty$ and some distribution P_α, $(\alpha = \overline{1,s})$ has a characteristic function that is integrable to some positive integer power ℓ, then the following propositions are true:

a) For almost every ω $(\omega\varepsilon\Omega)$ there is a natural number $N(\omega)$ such that when $n > N(\omega)$ is the conditional distribution $P_{S_{n_1}|\bar{\xi}_{1n}}$ has density $P_{S_{n_1}|\bar{\xi}_{1n}}(x)$ and

$$\sup_{x\epsilon R^k} |p_{S_{n_1}|\bar{\xi}_{1n}}(x) - \varphi_{R_0}(x)| \to 0 \ \ \textit{almost everywhere.}$$

b) Starting from the defined n_0 for the distribution of sum S_n a representation $P_{S_n} = P^{(1)}_{S_n} + P^{(2)}_{S_n}$ is valid (generally different from the expansion on absolutely continuous $P^{(\alpha c)}_{S_n}$ and singular $P^{(s)}_{S_n}$ parts with respect to the Lebesgue measure) such that, $P^{(s)}_{S_n}(R^k) \le P^{(2)}_{S_n}(R^k) \le C(\pi, P) \cdot n^{-1}$, where $C(\pi, P)$ is a constant depending on the parameters of the chain.

c) When $det(T_\mu) > 0$ and $p^{(1)}_{S_n}$ is $\mathrm{P}^{(1)}_{S_n}$ are the derivative of the Radon Nikodim measure: $P^{(1)}_{S_n}(x)$ with respect to the Lebesgue measure, is fulfilled the limit equality holds

$$\lim_{n\to\infty} \sup_{x\epsilon R^k} |p^{(1)}_{S_n}(x) - \varphi_{R_0+T_\mu}(x)| = 0$$

Proof. The validity of point a) follows from formulas (3) (according to[3]). Let's the conditions of the theorem be satisfied for the state $\alpha = 1$ of the chain. For number $\varepsilon > 0$,$(\varepsilon < min(\pi_1, 1 - \pi_1))$ let's introduce the events

$$B_n = \{|\frac{\nu_n(1)}{n} - \pi_1| < \varepsilon\} = \{n(\pi_1 - \varepsilon) < \nu_n(1) < n(\pi_1 + \varepsilon)\}, \ \ n = 1, 2, ...$$

Let's expand the distribution of the sum S_n as $P_{S_n}(\cdot) = EP_{S_n|\bar{\xi}_{1n}}(\cdot) = P^{(1)}_{S_n}(\cdot) + P^{(2)}_{S_n}(\cdot)$, where $P^{(1)}_{S_n}(\cdot) = EP_{S_n|\bar{\xi}_{1n}} I_{(B_n)}$, $P^{(2)}_{S_n}(\cdot) = EP_{S_n|\bar{\xi}_{1n}} I_{(\overline{B}_n)}$. Naturally $P^{(1)}_{S_n}(\cdot)$ starting from the number $n_0 = [(\pi_1 - \varepsilon)^{-1}] + 1$ is an absolutely continuous measure with respect to the Lebesgue measure. Let its Radon-Nikodim derivative be $p^{(1)}_{S_n}(x) = Ep_{S_n|\bar{\xi}_{1n}} I_{(B_n)}$.

The absolutely continuous measure would also be formed on those trajectories for that B_n does not fulfilled, therefore, according to (4)

$$P_{S_n}^{(\alpha c)}(R^k) \geq P_{S_n}^{(1)}(R^k),\ P_{S_n}^{(s)}(R^k) \leq P_{S_n}^{(2)}(R^k) = P(\bar{B}_n) \leq \frac{1}{\varepsilon^2}E|\frac{\nu_n(\alpha)}{n} - \pi_\alpha|^2 \leq \frac{c(\pi, P, \varepsilon)}{n}$$

We fix the trajectory $\overline{\xi}_{1n}$ and group separately those summands from (2), corresponding to the terms $\xi_i,\ i = \overline{1,n}$ of control sequence have obtained values $b_1, b_2, ..., b_s$ respectively. Their number is $\nu_n(\alpha),\ \alpha = \overline{1,s}$, respectively. Let's renumber the members of each group:

$$\tau_0(\alpha) = 0,\ \tau_m(\alpha) = min\{j|\tau_{m-1}(\alpha) < j \leq n;\ \xi_j = b_\alpha\},\ m = \overline{1, \nu_n(\alpha)},\ \alpha = \overline{1,s}.$$

Naturally that $\xi_{\tau_m(\alpha)} = b_\alpha,\ m = \overline{1, \nu_n(\alpha)},\ \alpha = \overline{1,s}$. Representations are valid

$$S_{n_1} = \frac{1}{\sqrt{n}}\sum_{\alpha=1}^{s} S_{n_1}(\alpha), \quad S_{n_1} = \sum_{m=1}^{\nu_n(\alpha)}[Y_{\tau_m(\alpha)} - \mu(\alpha)].$$

For each fixed state α, vectors $\xi_{\tau_m(\alpha)}, m = \overline{1, \nu_n(\alpha)}$ are equally distributed vectors with chain dependence. $EY_{\tau_m(\alpha)} = \mu(\alpha)$ and $cov(Y_{\tau_m(\alpha)} = R(\alpha)$. Therefore at $n \to \infty$

$$\sup_{A \epsilon C^k} |P_{\frac{S_{n_1}(\alpha)}{\sqrt{\nu_n(\alpha)}}|\bar{\xi}_{1n}}(A) - \Phi_{R(\alpha)}(A)| \to 0, \quad \textit{almost everywhere} \ \ \alpha = \overline{1,s}$$

Thereby

$$\sup_{x \epsilon R^k} |p_{\frac{S_{n_1}(1)}{\sqrt{\nu_n(1)}}|\bar{\xi}_{1n}}(x) - \varphi_{R(1)}(x)| I_{(B_n)} \to 0,\ \textit{almost everywhere}$$

For the number n $(n \geq n_0)$ let's designate $\tau_n = \sup_{x \epsilon R^k} |p_{S_{n_1}|\bar{\xi}_{1n}}(x) - \varphi_{R_0}(x)| I_{(B_n)}$. Let us show that $\tau_n \to 0$ almost everywhere. To this end, we decompose the sums S_{n_1} and R_0 into two parts:

$$S_{n_1} = \frac{1}{\sqrt{n}} S_{n_1}(1) + \frac{1}{\sqrt{n}}\sum_{\alpha=2}^{s} S_{n_1}(\alpha)$$

$$= \frac{1}{\sqrt{n}} S_{n_1}(1) + \frac{1}{\sqrt{n}}\overline{S}_{n_1}, R_0 = \pi_1 R(1) + \sum_{\alpha=2}^{s}\pi_\alpha R(\alpha) = \pi_1 R(1) + \bar{R}_0.$$

It is easy to determine that the conditional density of the sum $\frac{1}{\sqrt{n}} S_{n_1}(1)$ tends to the density $\varphi_{\pi_1 R(1)}$ and the asymptotic distribution of the sum $\frac{1}{\sqrt{n}}\bar{S_{n_1}}$ is Φ_{R_0}.

Starting from the number $n \geq n_0$ the value τ_n is bounded, so that $\lim_{n \to \infty} E\tau_n = 0$.

To prove point c) we apply the decomposition :

$$\sup_{x \epsilon R^k} |p_{S_n}^{(1)}(x) - \varphi_{R_0 + T_\mu}(x)| \leq \sup_{x \epsilon R^k} |Ep_{S_n|\bar{\xi}_{1n}}(x) I_{(B_n)} - \varphi_{R_0 + T_\mu}(x) \leq$$

$$\leq \sup_{x\epsilon R^k} |Ep_{S_n|\bar{\xi}_{1n}}(x)I_{(B_n)} - E\varphi_{R_0}(x - S_{n_2})I_{(B_n)}| + \sup_{x\epsilon R^k} |E\varphi_{R_0}(x - S_{n_2}) - \varphi_{R_0+T_\mu}(x)|$$
$$+ \sup_{x\epsilon R^k} |E\varphi_{R_0}(x - S_{n_2})I_{(\bar{B_n})}| \equiv I_1 + I_2 + I_3$$

The following estimate is valid

$$I_1 = \sup_{x\epsilon R^k} |Ep_{S_{n_1}|\bar{\xi}_{1n}}(x - S_{n_2})I_{(B_n)} - E\varphi_{R_0}(x - S_{n_2})I_{(B_n)}|$$
$$= E \sup_{x\epsilon R^k} |p_{S_{n_1}|\bar{\xi}_{1n}} - \varphi_{R_0}(x)|I_{(B_n)} = E\tau_n \stackrel{n\to\infty}{\longrightarrow} 0.$$

The summand I_2 would be represented as the following

$$I_2 = \sup_{x\epsilon R^k} |E\varphi_{R_0}(x - S_{n_2}) - \varphi_{R_0+T_\mu}(x)| = \sup_{x\epsilon R^k} |\int \varphi_{R_0}(x - y)(P_{S_{n_2}} - P_{T_\mu})(dy)|$$

According to equalities (3), it's limit is equal to zero. It's obvious that $I_3 \stackrel{n\to\infty}{\longrightarrow} 0$

□

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