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ON THE FLUID FLOW OVER THE RECTANGULAR AREAS

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Abstract. We consider 2D incompressible unsteady fluid flow over the rectangle and between two similar rectangles. The velocity components of the flow satisfy the nonlinear Navier - Stokes equations with the suitable initial-boundary conditions. We modify 2D NSE and find a new class of solutions. It is supposed that near sharp edges the velocity components are non-smooth and by the methods of mathematical physics we obtain novel exact solutions of NSE for the specific pressure.

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In the paper we study 2D incompressible viscous fluid flow over the rectangle and between two similar rectangles. The basic system of equations governing the flow is 2D Navier-Stokes Equations (NSE) with the appropriate initial-boundary conditions [1-7]. Our goal is to obtain the exact solutions of NSE in case of the specific pressure. First of all we modified NSE and considered the case of harmonic dynamical pressure. The exact solutions of the 2D NSE are obtained in some specific cases [1-7]. Here we found new non-trivial solutions.

Let us consider the area $D_t = \{D \times [0,T]; 0 \leq t \leq T\}$ with the boundary S_t, where D is the finite or infinite domain of xOy plane with the boundary S, and suppose, that the body force $\vec{F}$ has some potential i.e. there exists the function Φ for which $\vec{F} = grad\,\Phi$. In the area D_t we consider NSE with the equation of continuity

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\Delta u, \tag{1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\Delta v, \tag{2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3}$$

where $\vec{V}(u,v)$ is the velocity, $\vec{F}(F_x, F_y)$ is the body force, $P = p + p_0 - \rho\Phi$ is the dynamical pressure, p is the pressure, ρ is the density, ν is the kinematical viscosity of the fluid, p_0 is the certain constant, t is the time. The system (1), (2), (3) is considered with the initial-boundary conditions

$$u(x,y,0) = u_0(x,y),\ v(x,y,0) = v_0(x,y),\ \ u(x,y,t)|_S = 0,\ v(x,y,t)|_S = 0, \tag{4}$$

Below we will modify the system (1), (2), (3).

Modification of NSE. Let us take first order derivatives with respect to x of (1) and with respect to y of (2)

$$\frac{\partial}{\partial t}\Big(\frac{\partial u}{\partial x}\Big)+\Big(\frac{\partial u}{\partial x}\Big)^2+u\frac{\partial^2 u}{\partial x^2}+\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}+v\frac{\partial^2 u}{\partial y\partial x}=-\frac{1}{\rho}\frac{\partial^2 P}{\partial x^2}+\nu\Delta\frac{\partial u}{\partial x}, \tag{5}$$

$$\frac{\partial}{\partial t}\Big(\frac{\partial v}{\partial y}\Big)+\Big(\frac{\partial v}{\partial y}\Big)^2+v\frac{\partial^2 v}{\partial y^2}+\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}+u\frac{\partial^2 v}{\partial y\partial x}=-\frac{1}{\rho}\frac{\partial^2 P}{\partial y^2}+\nu\Delta\frac{\partial v}{\partial y}. \tag{6}$$

Taking into account (3) and summarize (5) and (6) one obtains

$$2\Big(\frac{\partial u}{\partial x}\Big)^2+2\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}=-\frac{1}{\rho}\Delta P\equiv F^*,\ 2\Big(\frac{\partial v}{\partial y}\Big)^2+2\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}=F^*. \tag{7}$$

From (7) we obtain the following nonlinear partial differential equations with respect to u and v

$$-2\frac{\partial^2 u}{\partial x^2}\Big(\frac{\partial u}{\partial y}\Big)^2+4\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y\partial x}-2\frac{\partial^2 u}{\partial y^2}\Big(\frac{\partial u}{\partial x}\Big)^2+F^*\frac{\partial^2 u}{\partial y^2}-\frac{\partial F^*}{\partial y}\frac{\partial u}{\partial y}=0, \tag{8}$$

$$-2\frac{\partial^2 v}{\partial y^2}\Big(\frac{\partial v}{\partial x}\Big)^2+4\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial y\partial x}-2\frac{\partial^2 v}{\partial x^2}\Big(\frac{\partial v}{\partial y}\Big)^2+F^*\frac{\partial^2 v}{\partial x^2}-\frac{\partial F^*}{\partial x}\frac{\partial v}{\partial x}=0. \tag{9}$$

The equations (8) and (9) do not contain time-derivatives and viscosity. Here we will consider the case $F^*=0$. In this case the equations (8),(9) become identical.

If $F^*=0$, solutions of the equation (1) and (2) satisfy the equations (8),(9), but not vice versa. It is clear that if u satisfies the equations (8),(9), then $-u$ is also the solution of (8),(9). Taking into account this result and by analyzing (1) we obtain the following theorem

Theorem. *If $F^*=0$ and the pressure is given by the formula $P=P^*(x,y)exp(\alpha t)$, (α is some constant), where $P^*(x,y)$ is a smooth function satisfying the condition $\frac{\partial P^*}{\partial x}=-\frac{\partial P^*}{\partial y}$, and if the smooth function $u^*(x,y)$ satisfies the equation*

$$\Delta u^*-\frac{\alpha}{\nu}u^*=\frac{1}{\nu\rho}\frac{\partial P^*}{\partial x},$$

conditions (4) and $\dfrac{\partial u^*}{\partial x}=\dfrac{\partial u^*}{\partial y}$, then the couple of functions $u=u^*(x,y)\,exp(\alpha t)$, $v=-u^*(x,y)\,exp(\alpha t)$ satisfies the equations (1), (2), (3) with the initial-boundary conditions (4).

Example 1. Let us consider the pair of functions

$$u=exp(-\alpha t)\,(sin(x+y)-C_0);\ v=-exp(-\alpha t)\,(sin(x+y)-C_0);|C_0|<1. \tag{10}$$

If $\alpha=\nu$ and $P=\alpha C_0(x-y)exp(-\alpha t)+A_0(t)$, where $A_0(t)$ is a continuous function of time, then the functions (10) are bounded solutions of the system (1),(2),(3) in the stripe

$$\arcsin C_0\le x+y\le\arcsin C_0+\pi.$$

Remark 1. The conditions of the Theorem are not necessary. For example, the pair of functions $u = exp(-\alpha t)\, sin(x - y); v = exp(-\alpha t)\, sin(x - y); \alpha = \nu;$ is the solution of (1), (2), (3), (4), in the stripe $x - \pi \leq y \leq x$ for the pressure $P = A_0(t)$.

Below, we consider 2D incompressible fluid flow in case of the non-smooth pressure.

Example 2. We now consider the fluid flow over the rectangle and between two rectangles. i.e. D is R^2 cut along the rectangle or the area between two similar rectangles. We assume that the pressure P and velocity components u, v are non-smooth functions and consider the cases:

1) $D = D_1$ is the rectangle of $x0y$ plane with the boundary $S_1 : \beta|x| + \beta|y| = B;\ B > 0$. Let us consider the functions

$$u_1 = sgn\, x\, (R_1\, exp\,(-\beta|x| - \beta|y| + B + \gamma t) - R_1\, exp\,(\gamma t)), \tag{11}$$

$$v_1 = -sgn\, y\, (R_1\, exp\,(-\beta|x| - \beta|y| + B + \gamma t) - R_1\, exp\,(\gamma t)), \tag{12}$$

where $R_1,\ \beta,\ \gamma$ are some real positive constants, $R_1,\ \beta,\ \gamma > 0$. The functions (11), (12) satisfy initial-boundary conditions (4) $u_1(x, y, t)|_{S_1} = 0$, $v_1(x, y, t)|_{S_1} = 0$, where $u_0(x, y) = u_1(x, y, 0)$, $v_0(x, y) = v_1(x, y, 0)$.

By direct verification we obtain:

If $\gamma = 2\beta^2\nu,\ P = p_0 + R_1\rho\gamma(|x| - |y|)exp\,(\gamma t)$ and $u_0(x, y) = u_1(x, y, 0)$, $v_0(x, y) = v_1(x, y, 0)$, then the functions (11), (12) are solutions of problem (1), (2), (3), (4). The velocity modulus will be given by the formula

$$|\vec{V}| = R_1\sqrt{2}\,|\,exp\,(-\beta|x| - \beta|y| + B + \gamma t) -\ exp\,(\gamma t))|. \tag{13}$$

2) Now, let us study the fluid flow between two rectangles $D_2 :\ B \leq \beta|x| + \beta|y| \leq B + \pi$. We consider the functions

$$u_2 = R_2\, sgnx\, exp\,(\gamma t) \times\ sin\,(-\beta|x| - \beta|y| + B), \tag{14}$$

$$v_2 = -R_2\, sgny\, exp\,(\gamma t)\ \times\ sin\,(-\beta|x| - \beta|y| + B), \tag{15}$$

$R_2,\ \beta,\ \gamma$ are some real positive constants, $R_2,\ \beta,\ \gamma > 0$.

The functions (14), (15) satisfy initial-boundary conditions (4)$u_2(x, y, t)|_{S_2} = 0$, $v_2(x, y, t)|_{S_2} = 0$, where $u_0(x, y) = u_1(x, y, 0)$ $v_0(x, y) = v_1(x, y, 0)$, S_2 is the boundary of D_2.

If $\gamma = -2\beta^2\nu$ and the pressure is given by the formula $P = A_0(t)$, where $A_0(t)$ is a continuous function of time, then (14) and (15) are solutions of problem (1), (2), (3), (4). The velocity modulus will be given by

$$|\vec{V}| = R_2\sqrt{2}\ exp(\gamma t)|\, sin\,(-\beta|x| - \beta|y| + B)|. \tag{16}$$

Remark 2. In example 2 the solutions of problem (1), (2), (3), (4) are non-smooth and the conditions of the Theorem do not satisfied.

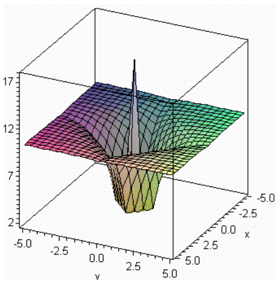

Figure 1: The velocity profile for (13) $\nu = 1,\ R_1 = 1,\ B = 1,\ \beta = 1,\ t = 1.$

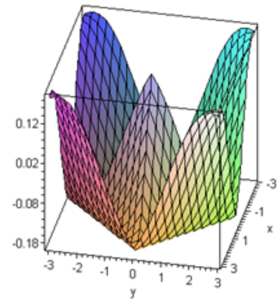

Figure 2: The velocity profile for (16) $\nu = 1,\ R_2 = 1,\ B = 1,\ \beta = 1,\ t = 1.$

Remark 3. The exact solutions of 3D NSE for the incompressible fluid flow over the prism and octahedron were obtained by the author in [5, 6], exact solutions for the creeping flows were obtained in [2-4].

Below (Fig. 1 and Fig. 2) profiles of velocities given by (13) and (16) are plotted
The velocity profile for (13)$\nu = 1,\ R_1 = 1,\ B = 1,\ \beta = 1,\ t = 1.$

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