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ON THE SOLUTIONS SPACE OF THE SPECIAL TYPE RIEMANN EQUATIONS

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Abstract. We investigate the relation between the coefficient α, β, γ of hypergeometric equations and the order p of the space L^p and give complete answer of the question, when the solutions belongs to L^p.

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1 Estimation of the solutions in singular points.

Consider the following differential equation with singular points $0, 1, \infty$

$$z(1-z)y''(z) + (\gamma - (\alpha+\beta+1))y'(z) - \alpha\beta y(z) = 0. \tag{1}$$

Below we investigate the following problem: *What condition should be satisfied α, β, γ and p, so that solutions of (1) belongs to L^p?*

Consider some particular cases.

It is known that the solution of differential equation (1) is a hypergeometric function, which has the form $F(\alpha, \beta, \gamma, z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k k!} z^k$, where $(\alpha)_n = \alpha(\alpha+1)(\alpha+2)...(\alpha+(n-1))$.

Example 1.

$$F(1,1,2,-z) = \sum_{k=0}^{\infty} \frac{k!k!}{(k+1)!k!}(-z)^k = \frac{1}{z}\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} z^{k+1} = \frac{ln(1+z)}{z}.$$

Suppose we have the curve $\Gamma = e^{it}$, $0 \le t \le 2\pi$. $ln(1+e^{i\pi}) = -\infty$, therefore $ln(1+e^{it})$ is not bounded on the interval $t \in [0, 2\pi]$, hence

$$\int_{\Gamma} \Big|\frac{ln(1+z)}{z} dz\Big| = \int_0^{2\pi} \Big|\frac{ln(1+e^{it}) i e^{it}}{e^{it}} dt\Big| = \int_0^{2\pi} |ln(1+e^{it})| dt = \infty,$$

therefore $F(1,1,2,-z) \notin L^1(\Gamma)$ and since $L^1(\Gamma) \supset L^2(\Gamma) \supset L^3(\Gamma) \supset L^4(\Gamma) \supset ...$, $F(1,1,2,-z) \notin L^p(\Gamma)$ $(1 \le p < \infty)$. In addition, since the space L^∞ has a maximum norm $(\max\limits_{t\in[0,2\pi]} |ln(1+e^{it})| = \infty)$, $F(1,1,2,-z) \notin L^\infty(\Gamma)$.

Example 2. If $\alpha = \beta = \gamma = 1$, then

$$F(1,1,1,z) = \sum_{k=0}^{\infty} \frac{(1)_k (1)_k}{(1)_k k!} z^k = \sum_{k=0}^{\infty} \frac{k! k!}{k! k!} z^k = \sum_{k=0}^{\infty} z^k = \frac{1}{1-z}, (|z| < 1).$$

If $\Gamma = Re^{it}$, where $|R| < 1$ and $0 \le t \le 2\pi$,

$$\int_\Gamma \frac{1}{1-z} dz = \int_0^{2\pi} \frac{iRe^{it}}{1-Re^{it}} dt = iR \int_0^{2\pi} \frac{e^{it}}{1-Re^{it}} dt = \int_R^R \frac{1}{1-u} du = 0.$$

($Re^{it} = u$, $t = 0 \implies u = R$, $t = 2\pi \implies u = R$, $Rie^{it}dt = du \implies dt = \frac{du}{Rie^{it}}$).

Since $|R| < 1$ is fixed $\implies Re^{it} \neq 1$ for all $0 \le t \le 2\pi \implies 1 - Re^{it} \neq 0$, $0 \le t \le 2\pi$, hence any $L^p(\Gamma)$ norm $\left(\int_\Gamma |f(z)| dz\right)^{\frac{1}{p}}$ $(1 \le p < \infty)$ is bounded, when maximum norm $p = \infty$ is also bounded. Therefore $F(1,1,1,z) = \frac{1}{1-z} \in L^p(\Gamma)$, when $|z| < 1$, $(1 \le p \le \infty)$.

Example 3. Simply can be obtained by induction higher order derivative formula for hypergeometric function.

$$\frac{d^n}{dz^n} F(\alpha, \beta, \gamma, z) = \frac{\alpha^n \beta^n}{\gamma^n} F(\alpha + n, \beta + n, \gamma + n, z).$$

$\frac{d^n}{dz^n} F(1,1,1,z) = \left(\frac{1}{1-z}\right)^{(n)} = \frac{P(z)}{(1-z)^{2^n}}$, where $deg(P(z)) < 2^n$, hence every derivative of $F(1,1,1,z)$ consists of some power (grater than 1) of $(1-z)$ in the denominator, but if we take again $\Gamma = Re^{it}$, where $|R| < 1$ and $0 \le t \le 2\pi$, similar to the previous reasoning we will get $\frac{d^n}{dz^n} F(1,1,1,z) = F(m,m,m,z) \in L^p(\Gamma), (1 \le p \le \infty)$.

If $\alpha \in \mathbb{Z}_- = \{..., -3, -2, -1\}$ (the same discussion works in the case of β and γ), then $(\alpha)_{|\alpha|+1} = 0$, hence $F(\alpha, \beta, \gamma, z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k k!} z^k = \sum_{k=0}^{|\alpha|} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k k!} z^k$. The last sum is finite, hence it is a polynomial with rational coefficients and with $|\alpha|$ degree, therefore for all sufficiently smooth, bounded curves $\Gamma \subset \mathbb{C}$ $F(\alpha, \beta, \gamma, z) \in L^p(\Gamma)$, $1 \le p \le \infty$. Hence, if α, β or $\gamma \in \mathbb{Z}_- = \{..., -3, -2, -1\}$, $F(\alpha, \beta, \gamma, z) \in L^p(\Gamma)$, $1 \le p \le \infty$.

The second linearly independent solution of equation (1) is

$$z^{1-\gamma} F(1-\gamma+\alpha, 1-\gamma+\beta, 2-\gamma, z) = \sum_{k=0}^{\infty} \frac{(1-\gamma+\alpha)_k (1-\gamma+\beta)_k}{(2-\gamma)_k k!} z^{k+1-\gamma}. \qquad (2)$$

which diverges in $\mathbb{C}$, when $z \in \{z : |z| = 1\}$.

If one of them from next conditions is satisfied $(\alpha, \beta, \gamma \in \mathbb{Z})$: $1+\alpha < \gamma$, $1+\beta < \gamma$, $2 < \gamma$, then from series (2) will survive only a finite sum, hence $z^{1-\gamma} F(1-\gamma+\alpha, 1-\gamma+\beta, 2-\gamma, z)$ will be a polynomial with finite $m \in \mathbb{N}$ degree and implies, for all sufficiently smooth and bounded curves $\Gamma \subset \mathbb{C}$ $z^{1-\gamma} F(1-\gamma+\alpha, 1-\gamma+\beta, 2-\gamma, z) \in L^p(\Gamma)$, $1 \le p \le \infty$.

2 On the Legendre equation. In this section we find the solutions of the Legendre equation

$$(1-z^2)y''(z)-2zy'(z)+k(k+1)y(z)=0, \tag{3}$$

as power series.

Since $z=0$ is not a singular point of differential equation (3), we are allowed to search solution of equation (3) in the form

$$y(z)=\sum_{n=0}^{\infty}a_nz^n. \tag{4}$$

So, we search coefficients a_n $(n=0,1,2,...)$ such that power series (4) representing the solution of equation (3).

If we use the rule of power series differentiation and input results in equation (3), we will get

$$(1-z^2)\sum_{n=2}^{\infty}a_nn(n-1)z^{n-2}-2z\sum_{n=1}^{\infty}a_nnz^{n-1}+k(k+1)\sum_{n=0}^{\infty}a_nz^n=0,$$

By the elementary transformations we will have the following expression.

$$\sum_{n=0}^{\infty}[a_{n+2}(n+2)(n+1)-a_nn(n-1)-2a_nn+k(k+1)a_n]z^n=0.$$

Therefore, since the system of polynomials $\{1,z,z^2,...,z^n,...\}$ is linearly independent, its coefficients have to be zero. It means that

$$a_{n+2}=\frac{a_n(n-k)(n+k+1)}{(n+2)(n+1)}. \tag{5}$$

For fixed k the second linearly independent solution can be found as follows. For simplicity assume $k=1$, the similar method can be used for other fixed k.

In the case $k=1$, we have already known that the first order Legendre polynomial $P_1(z)$ is a solution of differential equation (3). Based on that vector $(1,0)$ is linearly independent from vector $(0,1)$, if we set $a_0=1$ and $a_1=0$ next process gives us linearly independent solution of differential equation (3), because a_{2m+1} $(m=1,2,3,...)$ belongs a_1 as a multiplier and a_{2m} $(m=1,2,3,...)$ belongs a_0 as a multiplier.

$a_1=0$, by using (5) recursive relation we will have $a_{2m+1}=0$ $(m=1,2,3,...)$.

simply can be obtained by induction that $a_{n+2}=-\frac{1}{n+1}$, for all even n.

Therefore the second linearly independent solution of (3) is

$$y(z)=1-z^2-\frac{1}{3}z^4-\frac{1}{5}z^6-\frac{1}{7}z^8-\frac{1}{9}z^{10}-\frac{1}{11}z^{12}+...=\sum_{n=0}^{\infty}a_{2n}z^{2n}. \tag{6}$$

In addition $\sum_{n=0}^{\infty}|a_{2n}z^{2n}| = \sum_{n=0}^{\infty}|a_{2n}||z^{2n}| = \sum_{n=0}^{\infty}|a_{2n}||z|^{2n}$.

To prove that series (6) diverges when $|z| = 1$, consider the series $\sum_{n=1}^{\infty} b_n$, where $b_n = \frac{1}{10n+1}$.

$n = 0$, $a_0 = 1 \implies |a_0| \geq \frac{1}{10\times 0+1} = b_0$, suppose $a_{2n} \geq b_n$, when $n \geq 1$ and show that $a_{2n+2} \geq b_{n+1}$.

$$|a_{2n+2}| = \frac{|a_{2n}|(2n-1)}{2n+1} \geq \frac{1}{10\times n+1}\frac{2n-1}{2n+1} = \frac{2n-1}{20n^2+12n+1}$$

$$> \frac{2n-2}{20n^2+20n+2n} = \frac{n-1}{n(10n+11)} \geq \frac{1}{n(10n+11)} \geq \frac{1}{10(n+1)+1} = b_{n+1}.$$

Therefore, by induction we get $a_{2n} \geq b_n$ for all $n = 0, 1, 2, \ldots$. Hence,

$$\sum_{n=0}^{\infty}|a_{2n}| \geq \sum_{n=0}^{\infty}\frac{1}{10n+1} > \sum_{n=0}^{\infty}\frac{1}{10n+10} = \frac{1}{10}\sum_{n=0}^{\infty}\frac{1}{n+1} = \frac{1}{10}\sum_{m=1}^{\infty}\frac{1}{m}.$$

$\sum_{m=1}^{\infty}\frac{1}{m}$ is a divergent series. Therefore by comparison test $\sum_{n=0}^{\infty}|a_{2n}|$ diverges. But all terms $a_{2n} < 0$ except $a_0 = 1$ (finite number of elements does not affect the convergence of series), implies $\sum_{n=1}^{\infty} a_{2n} = -\sum_{n=1}^{\infty}|a_{2n}|$, therefore $\sum_{n=0}^{\infty} a_{2n}$ diverges. Now, when $|z| = 1$, $\sum_{n=0}^{\infty} a_{2n}z^{2n} = \sum_{n=0}^{\infty} a_{2n}cos(2nt) + i\sum_{n=0}^{\infty} a_{2n}sin(2nt)$. It is obvious, that either $cos^2(2nt) \geq \frac{1}{2}$ or $sin^2(2nt) \geq \frac{1}{2}$. Without loss of generality, for fixed $t_0 \in [0, 2\pi]$ suppose $cos^2(2nt_0) \geq \frac{1}{2}$ (same arguments work if $sin^2(2nt) \geq \frac{1}{2}$), then $|cos(2nt_0)| \geq \left|\frac{1}{\sqrt{2}}\right|$, hence either $cos(2nt_0) \geq \frac{1}{\sqrt{2}}$ or $cos(2nt_0) \leq -\frac{1}{\sqrt{2}}$. If $cos(2nt_0) \geq \frac{1}{\sqrt{2}}$, then $\sum_{n=0}^{\infty} a_{2n}cos(2nt_0) \geq \frac{1}{\sqrt{2}}\sum_{n=0}^{\infty} a_{2n}$, hence by the series comparison test, (6) series diverges at t_0. Now, if $cos(2nt_0) \leq -\frac{1}{\sqrt{2}}$, implies $-cos(2nt_0) \geq \frac{1}{\sqrt{2}}$. Therefore, by using this inequality and series comparison test, we will get divergence of series (6) taken with $-$ sign, hence we have divergence of series (6) at t_0. But $t_0 \in [0, 2\pi]$ was arbitrary, therefore series (6) diverges for all $t \in [0, 2\pi]$. Above we use the theorem, which says: complex series converges if and only if its both real and imaginary part converges.

Similarly can be found the second linearly independent solution of (1) equation for $k = 2, 3, 4, \ldots$ and its divergence can be established, when $|z| = 1$.

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