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ABOUT THE REGULAR T MATRIX OF THE TWO PARTICLE COULOMB SCATTERING *

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Abstract. The two-particle exact regular Coulomb T matrix of the continuous spectrum is represented in terms of the generalized functions, its main properties are formulated. Note that the aforementioned quantum mechanical function is defined in the $\mathbb{L}_2$ (Hilbert) space. This function is the exact solution of the two-particle homogeneous integral equation of the perturbation theory appropriate to the Coulomb T matrix.

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1 Motivation. It is known that the three charged particles problem of the continuous spectrum is unsolvable in the general case, but there are specific tasks that can be solved. See the following articles, for instance [1, 2].

Note that the solvability of the three-particle integral equations of the matrix T of the scattering reduces to the solvability of the corresponding two-particle equations ([3], pp. 121 - 128). In addition, in the few-body problems, the off-shell two-particle quantum-mechanical functions play an important role due to the presence of other particles.

According to the above, it is obvious how important the two-particle off-shell functions in few-body problems are.

An essential point of the stationary two-particle scattering theory is to find such solutions of the two-particle Schrödinger equation which have "free asymptotics" (solutions which turn to plane waves for large distances). To solve this problem it is necessary to show that the scattering (evolution) operator for the system under consideration exists and is defined in the Hilbert space. In the mentioned case, the perturbation theory can be used for the system of two particles of the continuous spectrum ([4], pp. 371 - 372, see also [5], pp. 39 - 42).

It is known that the assigned (stated) problem is solvable only for the specific class of potentials ([5], pp. 39-42). Unfortunately, the approach described above is not applicable to the Coulomb field ([4], pp. 181 - 197, see also [6], pp. 513 - 516).

Using the time-dependent Schrödinger theory, Dollard showed that the modified (renormalized) Coulomb scattering operator exists, and it is correct in the $\mathbb{L}_2$ space ([7], pp.

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27 - 30, see also [6], pp. 513 - 516). Let us note that the more rigorous mathematical formulation of the above result was proposed by Volker Enns [8].

As we see, the two-particle formalism of the Coulomb scattering theory predicts the existence of the two-particle Coulomb quantum-mechanical functions in the $\mathbb{L}_2$ space. At the same time, it is unclear how to build the above functions within the framework of the Lippmann-Schwinger formalism and what their structure is.

When we worked on this matter, we took into account the fact, that in the case of the Coulomb potential it is more convenient to study the integral representations of the considered quantum-mechanical functions than their integral equations.

It is known that the two-particle half-shell T matrix of the Coulomb scattering is expressed by the integral formula:

$$T_C^+\left(\vec{q},\vec{k}\right)=\int \exp\left(-i\vec{q}\vec{r}\right)V_C\left(r\right)\psi_C^+\left(\vec{r},\vec{k}\right)d\vec{r}, \tag{1}$$

where the function ψ_C^+ is the Coulomb function with the outgoing boundary condition.

Due to the weak convergence of the Coulomb potential V_C the infrared divergence appears in the expression (1), accordingly, the considered Coulomb T matrix (1) doesn't exist in a functional form. To avoid the above problem, the screened Coulomb potential is often used, which in the atomic unit system has the following form:

$$V_\alpha\left(r\right)=gr^{-1}\exp\left(-\alpha r\right), \tag{2}$$

where α is the screening (smooth cutoff) parameter ([3], 120 - 121).

Using the potential (2), one can define the exact Coulomb matrix T as follows:

$$T_C^+\left(\vec{q},\vec{k}\right)=\lim_{\alpha\to 0}\int\limits_0^{+\infty}\exp\left(-i\vec{q}\vec{r}\right)V_\alpha\left(r\right)\psi_C^+\left(\vec{r},\vec{k}\right)d\vec{r}. \tag{3}$$

It can be shown rigorously that, in its main aspects, representation (3) is compatible with the exact theory, since the cutoff in the above expression is removed ([6], pp. 513 - 516).

Instead of the potential (2) we prefer to use the following Coulomb function [9]:

$$\psi_{Cl}^+\left(\vec{r},\vec{k},\alpha\right)=\exp\left(-\alpha r\right)\psi_{Cl}^+\left(\vec{r},\vec{k}\right),\quad 0<\alpha\ll 1. \tag{4}$$

It is important to note, that the function (4) belongs to the set of Schwartz functions $\mathbb{S}$ of the class $\mathbb{C}$ of the complex plane.

From the calculational point of view the usage of the truncated wave function in the representation (3) is equivalent to the usage of potential (2), which allows us to solve the problem under consideration by the original approach [9].

Here we want to note the fact, that instead of the function (4) it would be better to use the following Coulomb distribution ([10], pp. 62 - 64):

$$\tilde{\psi}^{+}_{Cl}\left(\vec{r},\vec{k}\right)=\lim_{\alpha\to 0}\exp\left(-\alpha r\right)\psi^{+}_{Cl}\left(\vec{r},\vec{k}\right),\quad 0<\alpha\ll 1,$$

which quite naturally gives the interesting representation for us - from the computational point of view identical to the representation (3).

Below we will follow the nearest exact calculations, which are performed within the complex analysis and are described in the article [9].

2 Main results. Integrating the radial part of the representation (3), the radial T matrix of the Coulomb scattering, we get:

$$T^{+}_{Cl}(q,k)=H\left(\xi^2-\varepsilon^2\right)C_l\left(\gamma\right)V_{Cl}\left(q,k\right)-H\left(\xi^2-\varepsilon^2\right)\mathcal{O}\left(\varepsilon/k\right)+\mathcal{O}\left(\varepsilon/k\right),\qquad(5)$$
$$\xi=q-k,\quad 0<\varepsilon<<1,\quad \mathcal{O}\left(0/k\right)\equiv 0,$$

where $H\left(x\right)$ is the Heaviside Unit function, which is defined as follows:

$$H\left(\xi^2-\varepsilon^2\right)=\begin{cases}1 & \xi^2-\varepsilon^2>0,\\ 0 & \xi^2-\varepsilon^2\leqslant 0,\end{cases}=\begin{cases}1 & |\xi\,|>\varepsilon,\\ 0 & |\xi\,|\leqslant\varepsilon,\end{cases}\quad\varepsilon>0.$$

The coefficient $C_l\left(\gamma\right)$ and the Coulomb potential $V_{Cl}\left(q,k\right)$ see in [9].

By the formula (5), the off-shell Coulomb T matrix can be expressed as follows:

$$T^{+}_{Cl}(q,k)=C_l\left(\gamma\right)V_{Cl}\left(q,k\right),\quad q\neq k.$$

In addition, the expression (5) shows that the considered radial Coulomb T matrix in the vicinity of the energy shell ($q=k$) has the following asymptotic behavior:

$$T^{+}_{Cl}\left(q\to k\right)\to\mathcal{O}\left(\varepsilon/k\right),$$

which at the point $q=k$ turns into the following exact expression:

$$T^{+}_{Cl}\left(k,k\,\right)\equiv 0.$$

Let us note, that the function (5) expressed in terms of the generalized function $H\left(x\right)$ is the exact, regular, square integrable radial Coulomb function ([10], pp. 62 - 64).

The above calculations confirm that the limit (3) can be regarded as the unitarity restoration procedure in the definition (1) [9].

3 Conclusions. From the above, we can conclude that the regular Coulomb quantum-mechanical function $T^{+}_{C}\left(\vec{q},\vec{k}\right)$ belongs to the $\mathbb{L}_2$ (Hilbert) space. Accordingly, represents an orthogonal complete set of the continuous quantum-mechanical functions of the continuous spectrum. In addition, it is the exact solution of the two-particle homogeneous Coulomb T matrix integral equation of perturbation theory.

As we have already mentioned, the regular radial Coulomb T matrix is the complete set of the orthogonal continuous functions, which vanishes on the energy shell ($q = k$) and its small vicinity. The above property of the function under consideration shows that the dynamics of two charged interacting particles in the region of non-relativistic (low energy) kinematics on the energy shell is not considered.

Thus, according to newest theoretical calculations, an isolated system consisting of two interacting charged particles is experimentally unobservable. Indeed, in scattering experiments we never have any two charged particles. In fact, other charged particles are always present, which significantly influence the considered two-particle system due to the weak convergence of the Coulomb field.

Note, that the above results are also supported by the similar calculations performed for the Fourier transform of the Coulomb wave function.

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