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ON INVESTIGATION AND APPROXIMATE SOLUTION OF TWO SYSTEMS OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS ⋆

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Abstract. One-dimensional two models based on Maxwell's well-known system of nonlinear partial differential equations (SNPDE), describing the process of penetration of a magnetic field in a substance are considered. A unique solvability of the corresponding initial-boundary value problems and the convergence of finite-difference schemes are presented.

Keywords and phrases: Nonlinear partial differential equations, Maxwell's system, unique solvability, finite-difference scheme, convergence.

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Numerous scientific works, monographs and textbooks are devoted to the research of the SNPDE. One model of such type describing the process of electromagnetic field penetration in the substance is a well known system of Maxwell equations [1]:

$$\frac{\partial H}{\partial t} = -rot\left(\nu_m rot H\right), \tag{1}$$

$$\frac{\partial \Theta}{\partial t} = rot\left(\nu_m rot H\right), \tag{2}$$

where $H = (H_1, H_2, H_3)$ is a vector of the magnetic field, Θ is temperature, ν_m characterizes the electro-conductivity of the substance. As a rule, these coefficients are functions of the argument Θ. Equations (1) describe the process of diffusion of the magnetic field and equation (2) is change of the temperature at the expense of Joule heating. By the abovementioned Maxwell's system many important processes are described (see, e.g., [2]).

System (1), (2) does not take into account many physical effects. For a more thorough description, first of all it is desirable to take into consideration heat conductivity. In this case together with (1) instead of (2) the following equation is considered [1]

$$\frac{\partial \Theta}{\partial t} = \nu_m \left(rot H\right)^2 + div\left(k_m grad\,\Theta\right), \tag{3}$$

where k_m is a coefficient of heat conductivity. This coefficient is a function of Θ as well.

Note that, system (1), (2) can be reduced to the integro-differential form [3]. Many works are devoted to the investigation and numerical solution of initial-boundary value problems for (1), (2) and (1), (3) systems and for integro-differential models corresponding to (1), (2) (see, for example, [4]-[16] and references therein).

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Some aspects of the investigation and numerical solution of the one-dimensional version of system (1), (2) and (1), (3) in case of one-component magnetic field, are given, for example, in the works [2], [3], [5], [6], [11], [13].

Our aim is to consider one-dimensional version of systems (1), (2) and (1), (3) in case of the two-components magnetic field. Especially, unique solvability and the finite-difference schemes for some nonlinearities are constructed and investigated.

For most of SNPDEs it is very difficult to find exact solutions and there is no general solution available in a closed form. It is known that the exact solution for SNPDEs can be constructed in some particular cases.

In the domain $Q = (0;1) \times (0;\propto)$, let us consider the following initial-boundary value problem for Maxwell's type one-dimensional (1), (2) system:

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left(S^{\alpha}\frac{\partial U}{\partial x}\right), \quad \frac{\partial V}{\partial t} = \frac{\partial}{\partial x}\left(S^{\alpha}\frac{\partial V}{\partial x}\right), \tag{4}$$

$$\frac{\partial S}{\partial t} = S^{\alpha}\left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right], \tag{5}$$

$$U(0,t) = V(0,t) = 0, \quad U(1,t) = \psi_1 > 0, \quad V(1,t) = \psi_2 > 0, \tag{6}$$

$$U(x,0) = U_0(x), \quad V(x,0) = V_0(x), \quad S(x,0) = S_0(x). \tag{7}$$

Here $(x,t) \in Q$; $\alpha \in R$; ψ_1, ψ_2 are positive constants, and $U_0(x), V_0(x), S_0(x)$ are the given smooth functions and $U_0(0) = V_0(0) = 0$, $U_0(1) = \psi_1$, $V_0(1) = \psi_2$. Some qualitative and structural properties of solutions of (4)-(7) type problems are established in many works.

It is easy to check that if $U_0(x) = \psi_1 x, V_0(x) = \psi_2 x$ and $S_0(x) = S_0 = const > 0$, then when $\alpha \neq 1$ the solution of problem (4)-(7) is:

$$U(x,t) = \psi_1(x), \quad V(x,t) = \psi_2(x), \quad S(x,t) = \left[S_0^{1-\alpha} + (1 - \alpha(\psi_1^2 + \psi_2^2)t\right]^{\frac{1}{1-\alpha}}. \tag{8}$$

When $t_0 = S_0^{1-\alpha} \setminus [(\psi_1^2 + \psi_2^2)(\alpha - 1)]$ and $\alpha > 1$, from (8), it can be found that the function $S(x,t)$ is not bounded. The above example shows that (4) - (7) has no global solution at all. So, the solution of problem (4) - (7) with smooth initial and boundary conditions can be blown up at a finite time. The questions of unique solvability of some cases of these type problems are studied in the abovementioned literature and in a number of other works as well. Using [2] it is not difficult to prove the following statement.

Theorem 1. *If* $|\alpha| \leqslant 1/2$, *then the problem* (4)-(7) *has a unique solution.*

Note that if we add to (6) the following boundary conditions:

$$\left.\frac{\partial S}{\partial x}\right|_{x=0} = \left.\frac{\partial S}{\partial x}\right|_{x=1}, \tag{9}$$

then (U, V, S) defined by formulas (8) are also solutions of system with equations (4) and

$$\frac{\partial S}{\partial x} = S^{\alpha}\left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] + \frac{\partial^2 S}{\partial x^2}, \tag{10}$$

with (6), (7), (9) boundary and initial conditions. We conclude that for $\alpha > 1$, neither (4), (6), (7), (9), (10) problem has a global solution.

Introducing the following notation $E = S^{1/2}$ problem (4)-(7) takes the form:

$$\frac{\partial U}{\partial t} - \frac{\partial}{\partial x}\left(E^{2\alpha}\frac{\partial U}{\partial x}\right) = 0, \quad \frac{\partial V}{\partial t} - \frac{\partial}{\partial x}\left(E^{2\alpha}\frac{\partial V}{\partial x}\right) = 0, \tag{11}$$

$$\frac{\partial E}{\partial t} = \frac{1}{2}E^{2\alpha-1}\left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right], \tag{12}$$

$$U(0,t) = V(0,t) = 0, \quad U(1,t) = \psi_1, \quad V(1,t) = \psi_2, \tag{13}$$

$$U(x,0) = U_0(x), \quad V(x,0) = V_0(x). \tag{14}$$

Let us construct the usual grid on $[0,1] \times [0,T]$ and introduce the following notation [17]:

$$h = 1/M, \quad \tau = T/j, \quad x_i = ih, \quad t_j = j\tau, \quad u_i^j = u(x_i, t_j),$$

$$\overline{\omega} = \{x_i = ih, i = 0,1,...,M\}, \quad \omega_h^* = \{x_i = (i-1/2)\,h, i = 0,1,...,M\},$$

$$\omega_\tau = \{\tau_j = j\tau, i = 0,1,...,M\}, \quad \omega_{h\tau} = \overline{\omega}_\tau \times \omega_\tau^*, \quad \omega_{h\tau}^* = \omega_h^* \times \omega_\tau,$$

$$u_x = \frac{u_{i+1} - u_i}{h}, \quad u_{\overline{x}} = \frac{u_i - u_{i-1}}{h}, \quad u_i = u_i^{j+1}, \quad u_t = \frac{u_i - u_i^j}{h}.$$

Using usual notations and technique of building the finite-difference schemes (see, for example, [17]) let us construct an implicit finite-difference scheme for the problem (11)-(14):

$$u_t^j = (e^{2\alpha}u_{\overline{x}})_x, \quad v_t^j = (e^{2\alpha}v_{\overline{x}})_x, \tag{15}$$

$$e_t^j = \frac{1}{2}e^{2\alpha-1}(u_{\overline{x}}^2 + v_{\overline{x}}^2), \tag{16}$$

$$u_0^j = v_0^j = 0, \quad u_M^j = \psi_1, \quad v_M^j = \psi_2. \quad j = 0,1,...J, \tag{17}$$

$$u_i^0 = U_0(x_i), \quad v_i^0 = V_0(x_i), \quad e_i^0 = \left[S_0\left(x_{i+1/2}\right)\right]^{1/2}, \quad i = 0,1,...M-1, \tag{18}$$

where u and v functions are defined on the grid $\varpi_{h\tau}$ and the function e is defined on the grid $\omega_{h\tau}^*$. Here and below, unindexed values mean that the grid functions are taken at the point (x_i, t_{j+1}) or $(x_{i-1/2}, t_{j+1})$.

The approximations of the (15) - (18) scheme on the smooth solutions of problem (11)-(14) are of the order $O(\tau + h^2)$. The following statement is fair.

Theorem 2. *If* $|\alpha| \leqslant 1/2$*, then the scheme* (15)-(18) *converges to a solution of problem* (11)-(14) *in the grid functions space* L_2 *and the order of convergence is* $O(\tau + h^2)$.

Statements analogical to Theorems 1, 2 are true for problem (4),(6),(7),(9),(10) too.

The proof of the theorems presented in this article for wider nonlinearity, the description of algorithms for the approximate solution of the problems under discussion, and the presentation the results of the corresponding numerical experiments are planned in the next note.

REFERENCES

1. Landau, L., Lifschitz, E. Electrodynamics of Continuous Media, Course of Theoretical Physics. *Moscow*, 1957.
2. Dafermos, C. M., Hsiao, L. Adiabatic shearing of incompressible fluids with temperature dependent viscosity. *Quart. Appl. Math.,* **41** (1983/84), 45-58.
3. Gordeziani, D., Jangveladze, T., Korshija, T. Existence and uniqueness of a solution of certain nonlinear parabolic problems (Russian). *Differ. Uravn.,* **19** (1983), 1197-1207. *English translation: Differ. Equ.,* **19** (1983), 887-895.
4. Milani, A. Local in time existence for the complete Maxwell equations with monotone characteristic in a bounded domain. *Ann. Mat. Pura Appl.,* **131** (1982), 233-254.
5. Dzhangveladze, T. The difference scheme for one system of nonlinear partial differential equations (Russian). *Rep. Enlarged Sess. Semin. I. Vekua Inst. Appl. Math.,* **2** (1986), 40-43.
6. Abuladze, I.O., Gordeziani, D.G., Jangveladze, T.A., Korshija, T.K. Discrete models for a nonlinear magnetic-field-scattering problem with thermal conductivity (Russian). *Differ. Uravn.,* **22** (1986), 1119-1129. *English translation: Differ. Equ.,* **22** (1986), 1119-1129.
7. Cimatti, G. Existence of weak solutions for the nonstationary problem of the Joule heating of a conductor. *Ann. Mat. Pura Appl.,* **162** (1992), 33-42.
8. Yin, H.-M. Global solutions of Maxwell's equations in an electromagnetic field with a temperature-dependent electrical conductivity. *European J. Appl. Math.,* **5** (1994), 57-64.
9. Bien, M. Existence of global weak solutions for a class of quasilinear equations describing Joules heating. *Math. Meth. Appl. Sci.,* **23** (1998), 1275-1291.
10. Monk, P. Finite Element Methods for Maxwell's Equations. *Oxford University Press, New York,* 2003.
11. Sun, D., Manoranjan, V.S., Yin, H.-M. Numerical solutions for a coupled parabolic equations arising induction heating processes. *Discrete Contin. Dyn. Syst. Supplement,* (2007), 956-964.
12. Hong, M. C., Tonegawa, Y., Yassin, A. Partial regularity of weak solutions to Maxwell's equations in a quasi-static electromagnetic field. *Methods and Applications of Analysis,* **15** (2008), 205-222.
13. Gagoshidze, M., Jangveladze, T. On one nonlinear diffusion system. *Rep. Enlarged Sess. Semin. I. Vekua Inst. Appl. Math.,* **25** (2011), 39-43.
14. Jangveladze, T., Kratsashvili, M. Some properties of solution and finite difference scheme for one nonlinear partial differential model based on Maxwell system. *Mem. Differential Equations Math. Phys.,* **73** (2018), 83-92
15. Jangveladze, T., Kiguradze, Z., Neta, B. Numerical Solution of Three Classes of Nonlinear Parabolic Integro-Differential Equations, Elsevier, 2016, ACADEMIC PRESS, ISBN: 978-0-12-804628-9. Elsevier/Academic Press, Amsterdam, 2015.
16. Jangveladze, T. Investigation and numerical solution of nonlinear partial differential and integro-differential models based on system of Maxwell equations. *Mem. Differential Equations Math. Phys.,* **76** (2019), 1-118.
17. Samarskii, A.A. The Theory of Difference Schemes. *Marcel Dekker, Inc., New York,* 2001.

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