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THE ADOMIAN SERIES REPRESENTATION OF SOME CLASS OF BSDES

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Abstract. The representation of the solution of some Backward Stochastic Differential Equation as the Adomian infinite series is studied.

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Our aim is to prove an existence of the Adomian decomposition of the solution of the Backward Stochastic Differential Equation (BSDE)

$$dV_t = -\frac{1}{2}(\varphi_t^2 + \alpha\psi_t^2)dt - \gamma dA_t + \varphi_t dW_t + \psi_t dB_t,\ t \in [0,T],\ V_T = 0, \tag{1}$$

where $((W_t, B_t), \mathcal{F}_t)$ is a pair of independent brownian motions and $(A_t, \mathcal{F}_t)$ is a process of finite variation defined on the probability space $(\Omega, F, P, (\mathcal{F}_t))$.

Equations of such type are arising in mathematical finance and they are used to characterize optimal martingale measures [2].

The solution of BSDE (1) can be obtained by solving the equation

$$\frac{1}{2}\int_0^T (\varphi_t^2 + \alpha\psi_t^2)dt + \gamma A_T = c + \int_0^T \varphi_t dW_t + \int_0^T \psi_t dB_t. \tag{2}$$

with respect to (c, φ, ψ), where c is a constant.

Let $(V_t^{(n)},\ \varphi_t^{(n)},\ \psi_t^{(n)})$ be a sequence of solutions of

$$dV_t^0 = -\gamma dA_t + \varphi_t^{(0)} dW_t + \psi_t^{(0)} dB_t,\ V_1^{(0)} = 0,$$

$$dV_t^{(n)} = -\frac{1}{2}\sum_{k=0}^{n-1}(\varphi_t^{(k)}\varphi_t^{(n-k-1)} + \alpha\psi_t^{(k)}\psi_t^{(n-k-1)})dt + \varphi_t^{(n)} dW_t + \psi_t^{(n)} dB_t.$$

$$V_1^{(n)} = 0,\ n \geq 1$$

Then the triple

$$V_t = \sum_n V_t^{(n)},\ \varphi_t = \sum_n \varphi_t^{(n)},\ \psi_t = \sum_n \psi_t^{(n)}$$

will be a solution of (1), if the series is convergent [1].

Remark. If $A_t = \int_0^t a(s, W_s, B_s)ds$, then the solution of (1) is of the form $V_t = v(t, W_t, B_t)$, where $v(t, x, y)$ is decomposed as the series $\sum_n v^n(t, x, y)$ satisfying the system of PDEs

$$(\partial_t + \frac{1}{2}\Delta)v^0(t,x,y) + a(t,x,y) = 0, \quad v^0(T,x,y) = 0,$$

$$(\partial_t + \frac{1}{2}\Delta)v^n(t,x,y)$$

$$+\frac{1}{2}\sum_{k=0}^{n-1}(v_x^k(t,x,y)v_x^{n-k-1}(t,x,y) + \alpha v_y^k(t,x,y)v_y^{n-k-1}(t,x,y)) = 0,$$

$$v^n(T,x,y) = 0,\ n \geq 1.$$

To prove the convergence of series we need the following

Lemma. *Let* $(a_n)_{n\geq 0}$ *be a solution of the system*

$$a_0 = 1,\ a_{n+1} = \sum_{k=0}^{n} a_k a_{n-k} \tag{3}$$

and let $(b_n)_{n\geq 0}$ *be a non-negative solution of the system of inequalities*

$$b_0 \leq 1,\ b_{n+1} \leq \sum_{k=0}^{n} b_k b_{n-k}.$$

Then $a_n = \frac{1}{4n+2}\binom{2n+2}{n+1}$ *and* $b_n \leq a_n$, *for* $n \geq 0$.

Proof. We prove inequalities by induction. Assuming $b_k \leq a_k$ for all $k \leq n$ we get

$$b_{n+1} \leq \sum_{k=0}^{n} b_k b_{n-k} \leq \sum_{k=0}^{n} a_k a_{n-k} = a_{n+1},$$

since a_n and b_n are non-negatives.

On the other hand, for the series $u(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n$ from (3) we get equation $u(\lambda) = 1 + \lambda u^2(\lambda)$, with the roots $u(\lambda) = \frac{1}{2\lambda}(1 \pm \sqrt{1-4\lambda})$. The equality $u(\lambda) = \frac{1}{2\lambda}(1+\sqrt{1-4\lambda})$ is impossible, since decomposition of the right hand side is starting from the term $\frac{1}{\lambda}$. Therefore, equality $a_n = \frac{1}{4n+2}\binom{2n+2}{n+1}$ follows from the Taylor expansion of $1-\sqrt{1-4\lambda}$, since

$$u(\lambda) = \frac{1}{2\lambda}(1-\sqrt{1-4\lambda})$$

$$= -\frac{1}{2}\sum_{n\geq 1}\frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-n+1)}{n!}(-4)^n\lambda^{n-1}$$

$$= \frac{1}{2}\sum_{n\geq 1}\frac{(2-1)\cdots(2n-2-1)}{2^n n!}4^n\lambda^{n-1}$$

$$= \frac{1}{2}\sum_{n\geq 1}\frac{(2n-3)!!}{n!}2^n\lambda^{n-1} = \frac{1}{2}\sum_{n\geq 1}\frac{1}{2n-1}\binom{2n}{n}\lambda^{n-1}.$$

We use notation $|M|_{BMO} = \inf\{C : E^{\frac{1}{2}}(\langle M\rangle_T - \langle M\rangle_\tau|\mathcal{F}_\tau) \leq C\}$ for BMO-norms of martingales, $|A|_\omega = \inf\{C : var_t^T(A) \leq C\}$ for norms of finite variation processes and $A \cdot M$ for stochastic integrals. □

Theorem. *The series $\sum_{n\geq 0} \varphi^{(n)} \cdot W + \psi^{(n)} \cdot B$ is convergent in BMO-space, if γ is small enough and the sum of series is a solution of equation (2).*

Proof. For a martingale $L_t = E(A_T|\mathcal{F}_t)$ we have estimation $|L|_{BMO} \leq 2|A|_\omega$ [2]. Thus

$$|\varphi^{(0)} \cdot W + \psi^{(0)} \cdot B|_{\text{BMO}} \leq 2\gamma|A|_\omega,$$

$$\max(|\varphi^{(n+1)} \cdot W|_{\text{BMO}}, |\psi^{(n+1)} \cdot B|_{\text{BMO}}) \leq |\varphi^{(n+1)} \cdot W + \psi^{(n+1)} \cdot B|_{\text{BMO}}$$

$$\leq \operatorname*{ess\,sup}_\tau \sum_{k=0}^{n} E\Big(\int_\tau^T \Big|\sum_k^n \Big(\varphi_s^{(k)}\varphi_s^{(n-k)} + \alpha\psi_s^{(k)}\psi_s^{(n-k)}\Big)ds||\mathcal{F}_\tau\Big)$$

$$\leq \sum_{k=0}^{n} \operatorname*{ess\,sup}_\tau E^{\frac{1}{2}}\Big(\int_\tau^T |\varphi_s^{(k)}|^2 ds|\mathcal{F}_\tau\Big)E^{\frac{1}{2}}\Big(\int_\tau^T |\varphi_s^{(n-k)}|^2 ds|\mathcal{F}_\tau\Big)$$

$$+|\alpha| \sum_{k=0}^{n} \operatorname*{ess\,sup}_\tau E^{\frac{1}{2}}\Big(\int_\tau^T |\psi_s^{(k)}|^2 ds|\mathcal{F}_\tau\Big)E^{\frac{1}{2}}\Big(\int_\tau^T |\psi_s^{(n-k)}|^2 ds|\mathcal{F}_\tau\Big)$$

$$\leq \sum_k^n |\varphi^{(k)} \cdot W|_{\text{BMO}}|\varphi^{(n-k)} \cdot W|_{\text{BMO}} + |\alpha||\psi^{(k)} \cdot B|_{\text{BMO}}|\psi^{(n-k)} \cdot B|_{\text{BMO}}$$

$$\leq (1 + |\alpha|) \sum_{k=0}^{n} |\varphi^{(k)} \cdot W + \psi^{(k)} \cdot B|_{\text{BMO}}|\varphi^{(n-k)} \cdot W + \psi^{(n-k)} \cdot B|_{\text{BMO}}.$$

For $b_n = (1 + |\alpha|)^n|\varphi^{(n)} \cdot W + \psi^{(n)} \cdot B|_{BMO}|\varphi^{(0)} \cdot W + \psi^{(0)} \cdot B|_{BMO}^{-n-1}$ we have $b_0 = 1, b_{n+1} \leq \sum_k^n b_k b_{n-k}$. Using the lemma we get

$|\varphi^{(n)} \cdot W + \psi^{(n)} \cdot B|_{\text{BMO}} \leq a_n(1 + |\alpha|)^{-n}|\varphi^{(0)} \cdot W + \psi^{(0)} \cdot B|_{BMO}^{n+1} \leq a_n(1 + |\alpha|)^{-n}|2\gamma A|_\omega^{n+1}$.

Since $\overline{\lim}_{n\to\infty} \sqrt[n]{a_n} = \overline{\lim}_{n\to\infty} \sqrt[n]{\frac{1}{2n+1}\binom{2n+2}{n+1}} = \overline{\lim}_{n\to\infty} \sqrt[n]{\frac{(2n)!}{n!n!}} = \overline{\lim}_{n\to\infty} \sqrt[n]{\frac{(2n)^{2n}}{n^{2n}}} = 4$, the series is convergent, when $\gamma < \frac{1+|\alpha|}{8|A|_\omega}$. □

R E F E R E N C E S

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