Reports of Enlarged Sessions of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 36, 2022

LAMÉ CURVES AND RVACHEV'S R-FUNCTIONS *

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Abstract. Gielis transformations are a generalization of Lamé curves. To combine domains, we can make use of the natural alliance between Lamé's work and Rvachevs R-functions. A logical next step is the extension to n-valued logic defining different partitions.

Keywords and phrases: Lamé curves, Gielis Transformations, R-functions, n-valued logic

AMS subject classification (2010): 53A04, 53A05, 03B50.

1 Introduction. Superellipses, special instances of the family of curves named after Gabriel Lamé, are defined as

$$|x/a|^n + |y/b|^n = R^n(1)$$

Lamé curves include all conic sections as special cases, and they can serve as basis functions for superquadrics, including superellipsoids, supertoroids and superhyperboloids. Similarly, the boundary of superellipsoids is defined as:

$$|x/a|^n + |y/b|^n + |z/c|^n = R^n(2)$$

Gielis transformations (GT) are a generalization of Lamé curves, and hence of the circle and the Pythagorean theorem. Its origin is in modeling natural shapes and phenomena, and this family of curves, surfaces and bodies can be used in CAD/CAE/CAM [**?**].

2 Constructive Solid Geometry. CSG applies Boolean operations to generate complex shapes, consisting of a variety of individual solids, often called primitives. If for any solid S, the set of its interior points is denoted by I, the set of its boundary points by B and the set of its exterior points by T, then $I \cup B \cup T = E^3$ and $I \cap B = B \cap T = I \cap T = \varnothing$. A continuous function $f(P)$, non-negative for every P in E^3 will be called a defining function for a solid S with $f(P) < 1$ when P belongs to I, $f(P) = 1$ when P belongs to B, and $f(P) > 1$ when P belongs to T. Thus, for any solid S with $f(P)$ as a defining function, the surface equation is $f(P) = 1$ [2].

In the combination of shapes and primitives one can choose among a variety of Boolean functions, but any Boolean function $f : \{0,1\}^n \to \{0,1\}$ can be expressed by $\{\vee, \wedge, \neg\}$, in other words the system $\{\vee, \wedge, \neg\}$ is complete. The basic algebra for classical logic is a two-element Boolean algebra $(\{0,1\}, \vee, \wedge, \neg, 0, 1,$ where $\neg x = 1 - x; x \vee y = \max(x, y)$ and $x \wedge y = \min(x, y)$.

*Dedicated to Yohan Fougerolle

However, it can be done also by $\{\wedge, \neg\}$, since $x \vee y = \neg(\neg x \wedge \neg y)$. So, the system $\{\wedge, \neg\}$ is also complete. In fact $\vee$ and $\wedge$ are interchangeable because of De Morgans law $\neg(x \vee y) = (\neg x \wedge \neg y)$.

In order to perform the relevant studies in CAD/CAE (and engineering in general) the max and min functions must be approximated by means of suitable functions depending on a parameter which can be used to control the degree of smoothing. Differentiable approximating functions can be used, provided that the defining functions involved in the intersection and union operations are themselves differentiable. Approximating functions chosen to be substituted for max and min functions may be functions such as $I_p(f_1, f_2, ..., f_n) = (f_1^p + f_2^p + ... + f_n^p)^{1/p}$, and $U_p(f_1, f_2, ..., f_n) = (f_1^{-p} + f_2^{-p} + ... + f_n^p)^{-1/p}$ where p is a positive real number [2].

3 R-functions and geometric algebra. R-functions, named after the Ukranian mathematician V.L. Rvachev (1926-2005), give a geometrical method for constructing functions that exactly represent virtually any geometric shape of interest in engineering [3] One can build complex assemblies, consisting of many primitives define by implicit equations, into one single equation. Interpreting positive values as true and negative values as false, an R-function is transformed into a "companion" Boolean function. For instance, the R-function $f(x, y) = \min(x, y)$ is one possible friend of the logical conjunction (AND). This methodology is also known as semi-analytic geometry [4].

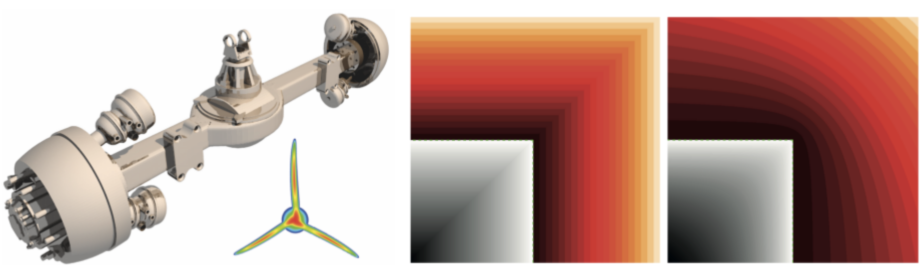

Figure 1: Rear truck axis, turbine and potential fields, defined by $R_{\alpha=1}$ and $R_{p=2}$ [7]

R-functions are generally described by the sign function, but with superellipses the absolute values play this role. One example: a rectangular plate (domain Ω_1) may be joined with a circular one (domain Ω_2) and this union is then described by the Boolean operation union (or max) of Ω_1 and Ω_2. This can be achieved by using two superellipses $\Omega_1 : 1 - \max(\frac{x}{a}, \frac{y}{b}) = 0$ (with $a = b$) and $\Omega_2 : 1 - (x^2 + y^2) = 0$, respectively. Hence, $\Omega = \Omega_1 \cup \Omega_2$. Y. Fougerolle [5] gives the example of turbine blades and a rear axis of a truck, consisting of 256 primitives (Figure 1), defined by a single equation, and uses the same for the reverse process: scanning a rear axis and reconstructing it with supershapes [6].

The set of all R-functions that have the same logic companion function is called a branch of the set of R-functions. Since the number of distinct logic functions of n arguments is 2^n, there are also 2^{2^n} distinct branches of R-functions of n arguments. The set of R-functions is in principle, infinite. However, for applications, it is not necessary to know all R-functions; it is needed to be able to construct R- functions that belong to a specified branch.

4 R-functions, Lamé and GML bodies. The power of R-functions lies in the potential to ensure differentiability to any desired order on the complex domain. Using different R-functions one can define different potential fields (Figure 1). The R_α functions corresponding to logical conjunction $f = x_1+x_2-\sqrt{x_1^2 + x_2^2 - 2\alpha x_1 x_2}$ and logical disjunction $f = x_1+x_2+\sqrt{x_1^2 + x_2^2 - 2\alpha x_1 x_2}$, as well as the R_p functions $f = x_1+x_2- [x_1^p+x_2^p]^{1/p}$ and $f = x_1 + x_2 + [x_1^p + x_2^p]^{1/p}$ satisfy the desired properties.

In a geometrical way, to a triangle with sides x_1 and x_2, a side is added or subtracted which is a Lamé-Minkowski distance, with the Euclidean distance for $p = 2$. If one considers the $R_{p=2}$ function, it can be rewritten in a trigonometrical form [7]. This combination also gives the possibility to a continuous transformation between logical functions conjunction, disjunction and difference.

One of the most interesting applications is in the study self-intersecting curves, for example, defined by Gielis curves with m a rational number and in Generalized Möbius-Listing bodies [8]. This also led to the first algorithm, capable of detecting self-intersecting curves and domains in vision and pattern recognition [9].

5 R-functions for different partitions of space. A logical next step is the generalization from 2-valued (Boolean) logic to n-valued logic defining different partitions. In CSG boundary representations should include "thickness" (annuli, shells, membranes,). Another partition may be true/false/possibly true, to deal with uncertainties, for example, in complex systems like immunology [10]. The notion of R-functions is a special case of a more general concept of R-mapping that is associated with qualitative k-partitions of arbitrary domains and multi-valued logic functions.). Then for R-functions we need a 3-valued logic (or an n-valued logic for n partitions). Such partition may also indicate true/false/possible true (or current state unknown).

Let $S_2 = \{0, 1/2, 1]$ be endowed with the following operations: $x \oplus y = \min(1, x+y), x \otimes y = \max(0, x+y-1), \neg x = 1-x, 0, 1/2, 1$, that becomes an algebra $(S_2, \oplus, \otimes, \neg x, 0, 1/2, 1)$, where $(S_2, \oplus, \otimes, \neg x, 0, 1)$ is an MV_3-algebra [11]. The algebra S_2 is isomorphic to the algebra $E_3 = (\{-1, 0, 1\}, \oplus, \otimes, \neg, -1, 0, 1)$ of type $(2, 2, 1, 0, 0, 0)$, where $x \otimes y = max(-1, x + y-1)$, $\neg$ is changing of sign and $x \oplus y = \neg(\neg x \otimes \neg y)$, by the mapping $-1 \mapsto 0, 0 \mapsto \frac{1}{2}, 1 \mapsto 1$.

We have MV_3-algebra with constants: $(\{-1, 0, 1\}, \otimes, \neg, -1, 0, 1)$ which is functionally equivalent to 3-valued Post algebra [11].

Therefore we have

Theorem. *The number of branches of n-ary R-functions is equal to* 3^{3^n}.

The extension to different partitions is a subject of ongoing research. A further area of research is whether the method of Fourier projection to solve boundary value problems on all normal polar domains in 2D and 3D can be extended to compositions of domains [12].

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Received 23.05.2022; accepted 20.07.2022.

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