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ON ONE SYSTEM OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. One system of nonlinear partial differential equations is considered. Uniqueness and stability of solution of initial-boundary value problem is studied.

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The mathematical description of many processes is carried out partially by differential equations and their systems. The types of models of the system of nonlinear equations discussed in this article are partially derived from the description of real diffusion processes (see, for instance, [1]-[7], [10] and references therein), and on the other hand, in the generalization of well-known equations and systems of equations, the study of which is devoted to many scientific papers (see, for instance, [8], [9] and references therein).

Our article discusses one system of nonlinear partial differential equations [3]. The uniqueness and stability of the solution of the initial-boundary value problem are investigated.

In the rectangle $\{(x,t)|x \in [0,1]; t \in [0,T]\}$, where $T = const > 0$, consider the following initial-boundary value problem:

$$\frac{\partial U}{\partial t} + \frac{\partial^2}{\partial x^2}\left[(1+V)\frac{\partial^2 U}{\partial x^2}\right] = f(x,t), \tag{1}$$

$$\frac{\partial V}{\partial t} = \left(\frac{\partial^2 U}{\partial x^2}\right)^2, \tag{2}$$

$$U(0,t) = U(1,t) = 0, \quad \frac{\partial^2 U}{\partial x^2}(0,t) = \frac{\partial^2 U}{\partial x^2}(1,t) = 0, \tag{3}$$

$$U(x,0) = U_0(x), \quad V(x,0) = 0. \tag{4}$$

Multiply equation (1) by U and integrate the obtained equation by $[0,1]$. If use the formula of integration by parts twice and boundary conditions (3), we get

$$\frac{1}{2}\int_0^1 \frac{\partial U^2}{\partial t}dx + \int_0^1 \left[(1+V)\frac{\partial^2 U}{\partial x^2}\right]\frac{\partial^2 U}{\partial x^2}dx = \int_0^1 f(x,t)U dx.$$

Since $V(x,t) \geq 0$, we use the inequality $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$, we obtain:

$$\frac{1}{2}\int_0^1 \frac{\partial U^2}{\partial t}dx + \int_0^1 \left(\frac{\partial^2 U}{\partial x^2}\right)^2 dx \leq \frac{1}{2}\int_0^1 f^2 dx + \frac{1}{2}\int_0^1 U^2 dx.$$

The condition (3) allows us to use the Poincaré-Friedrichs inequality [8]

$$\int_0^1 U^2 dx \leq \int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 dx \leq \int_0^1 \left(\frac{\partial^2 U}{\partial x^2}\right)^2 dx,$$

so we have:

$$\frac{1}{2}\int_0^1 \frac{\partial U^2}{\partial t}dx + \int_0^1 \left(\frac{\partial^2 U}{\partial x^2}\right)^2 dx \leq \frac{1}{2}\int_0^1 f^2 dx + \frac{1}{2}\int_0^1 \left(\frac{\partial^2 U}{\partial x^2}\right)^2 dx,$$

$$\frac{d}{dt}\|U\|^2 \leq \|f\|^2.$$

After integrating over t, taking into account the initial condition (4), we have:

$$\|U\|^2 \leq \int_0^t \|f(\tau)\|^2 d\tau + \|U_0\|^2.$$

These inequalities and equations (2) allow rating V on $f(x,t)$ and $U_0(x)$.

The resulting estimates means the stability of the solution of problem (1)-(4) with respect to the right hand side and the initial conditions.

Now, let's prove the uniqueness of the solution for (1)-(4). Suppose that (U_1, V_1) and (U_2, V_2) are two solutions of problem (1)-(4). For $W = U_1 - U_2, Z = V_1 - V_2$ we have:

$$\frac{\partial W}{\partial t} + \frac{\partial^2}{\partial x^2}\left[\frac{\partial^2 W}{\partial x^2} + V_1\frac{\partial^2 U_1}{\partial x^2} - V_2\frac{\partial^2 U_2}{\partial x^2}\right] = 0, \tag{5}$$

$$\frac{\partial Z}{\partial t} = \left(\frac{\partial^2 U_1}{\partial x^2}\right)^2 - \left(\frac{\partial^2 U_2}{\partial x^2}\right)^2, \tag{6}$$

$$W(0,t) = W(1,t) = 0, \quad \frac{\partial^2 W}{\partial x^2}(0,t) = \frac{\partial^2 W}{\partial x^2}(1,t) = 0, \tag{7}$$

$$W(x,0) = Z(x,0) = 0, \quad V_1(0,t) = V_2(1,t) = 0. \tag{8}$$

Let us multiply equation (5) by W and integrate the obtained equation by $[0,1]$. If we use the formula of integration by parts twice, boundary conditions (7), we get:

$$\frac{1}{2}\frac{d}{dt}\|W\|^2 + \left(\frac{\partial^2 W}{\partial x^2}\right)^2 + \int_0^1 \left(V_1\frac{\partial^2 U_1}{\partial x^2} - V_2\frac{\partial^2 U_2}{\partial x^2}\right)\left(\frac{\partial^2 U_1}{\partial x^2} - \frac{\partial^2 U_2}{\partial x^2}\right) dx = 0.$$

Use the easily verifiable inequality

$$(ca - db)(a - b) \geq \frac{1}{2}(c - d)(a^2 - b^2),$$

where assuming that

$$a = \frac{\partial^2 U_1}{\partial x^2}, \quad b = \frac{\partial^2 U_2}{\partial x^2} \quad c = V_1, \quad d = V_2.$$

Since

$$V_1(x,t) = \int_0^t \left(\frac{\partial^2 U_1}{\partial x^2}\right)^2 d\tau, \quad V_2(x,t) = \int_0^t \left(\frac{\partial^2 U_2}{\partial x^2}\right)^2 d\tau.$$

We have

$$\frac{d}{dt}||W||^2 + \int_0^1 \int_o^t \left[\left(\frac{\partial^2 U_1}{\partial x^2}\right)^2 - \left(\frac{\partial^2 U_2}{\partial x^2}\right)^2\right] d\tau \cdot \left(\frac{\partial^2 U_1}{\partial x^2} - \frac{\partial^2 U_1}{\partial x^2}\right) dx \leq 0.$$

From (6):

$$Z(x,t) = \int_0^t \left[\left(\frac{\partial^2 U_1}{\partial x^2}\right)^2 - \left(\frac{\partial^2 U_2}{\partial x^2}\right)^2\right] d\tau,$$

Finally, we get

$$\frac{d}{dt}\|W\|^2 + \frac{1}{2}\frac{d}{dt}||Z||^2 \leq 0.$$

After integrating over t and taking into account the initial condition (8), we obtain

$$\|W\|^2 + \frac{1}{2}\|Z\|^2 \leq 0,$$

which proves the uniqueness of the solution of problem (1)-(4).

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