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ON THE PROPAGATION OF AN EXPLOSIVE SHOCK WAVE IN A HOMOGENEOUS GRAVITATING THREE-AXIS GAS ELLIPSOID OF ROTATION

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Abstract. In this paper, we consider a non-self-similar problem of a central explosion of a homogeneous gaseous three-axis ellipsoid located in its own gravitational field and solid-body rotating around an z axis with a constant ω angular velocity. As initial data, the exact solution of the problem of stationary solid-state rotation of a homogeneous three-axis gas Jacobi ellipsoid is considered. It is assumed that at the initial moment of time a central explosion occurs with the release of finite energy. In this case, a diverging shock wave is formed in the center (a discontinuity surface of the first kind of the unknown functions). To solve the problem, we used the previously proposed asymptotic method of a thin shock layer. Zero approximations are found for the singular asymptotic expansion of the law and the velocity of the medium behind the shock wave, as well as the thermodynamic characteristics of the gravitating gas.

Keywords and phrases: Rotating three-axis gas ellipsoid of Jacobi, gravitational field, shock wave, singular asymptotic expansion.

AMS subject classification (2010): 97M50, 83F05, 76U05.

1 Introduction. To solve many problems of astrophysics, it is necessary to study the dynamics of gaseous bodies interacting with a gravitational field [1–7]. Numerous observations show that stars rotate.The study of rotating gravitating gaseous bodies has a long history and originates in the classical works of Newton, Maclaurin, Jacobi, Liouville, Dirichlet, Dedekind, Riemann, Poincar, Lyapunov and others [8].

2 The system of equations in partial derivatives. When considering explosive phenomena in gravitational bodies, it is convenient to use the equations of three-dimensional motion in Lagrangian coordinates, which are

$$\begin{gathered}
\frac{\partial^2 r}{\partial t^2} - r\Big[\Big(\frac{\partial\theta}{\partial t}\Big)^2 + \sin^2\theta\Big(\frac{\partial\varphi}{\partial t}\Big)^2\Big] = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{\partial\Phi}{\partial r},\\
\frac{\partial}{\partial t}\Big[r^2\frac{\partial\theta}{\partial t}\Big] - \frac{r^2\sin 2\theta}{2}\Big(\frac{\partial\varphi}{\partial t}\Big)^2 = -\frac{1}{\rho}\frac{\partial p}{\partial\theta} + \frac{\partial\Phi}{\partial\theta},\\
\frac{\partial}{\partial t}\Big[r^2\sin^2\theta\frac{\partial\varphi}{\partial t}\Big] = \sin\theta\Big[-\frac{1}{\rho}\frac{\partial p}{\partial\varphi} + \frac{\partial\Phi}{\partial\varphi}\Big],\\
\rho r^2\sin\theta\det\Big|\ \frac{\partial x^i}{\partial\xi^i}\Big| = \frac{\sin\hat\theta}{4\pi},\ p = p(m,\hat\theta,\hat\varphi,t) = (\gamma-1)f(m,\hat\theta,\hat\varphi)\rho^\gamma,
\end{gathered} \tag{1}$$

Φ is the potential of the gravitational field,

$r = r(m, \hat{\theta}, \hat{\varphi}, t)$, $\theta = \theta(m, \hat{\theta}, \hat{\varphi}, t)$, $\varphi = \varphi(m, \hat{\theta}, \hat{\varphi}, t)$, $p = p(m, \hat{\theta}, \hat{\varphi}, t)$, $\rho = \rho(m, \hat{\theta}, \hat{\varphi}, t)$ are unknown functions, $m = \frac{4\pi}{3}\rho_0 \hat{r}^3$, $\rho_0 = const$, $m, \hat{\theta}, \hat{\varphi}$, are Lagrangian coordinates, $f(m, \hat{\theta}, \hat{\varphi})$ is the function related to the distribution of entropy by Lagrangian coordinates.

3 Problem statement. Let a homogeneous gas tree-axis ellipsoid in its own gravitational field rotate solid about the axis z with ω angular velocity. Suppose that at a moment in time an explosion occurs with the release of E energy. In this case, a divergent shock wave forms in the center. It is necessary to determine the law and the velocity of the medium behind the shock wave, as well as the thermodynamic characteristics of the medium.

The conditions on the discontinuity surface of the first kind, permitted with respect to the parameters of the gas behind the shock wave, noted with index 2, have the form:

$$
\begin{gathered}
\rho_2 = \frac{\gamma+1}{\gamma-1}\rho_1 \Lambda^{-1}, \; p_2 = \frac{2}{\gamma+1}\rho_1 \frac{[\frac{\partial R}{\partial t} - (\frac{\partial r}{\partial t} - \frac{\partial R}{\partial \theta}\frac{\partial \theta}{\partial t} - \frac{\partial R}{\partial \varphi}\frac{\partial \varphi}{\partial t})_1]^2}{B^2} - \frac{\gamma-1}{\gamma+1}p_1, \\
B \equiv \sqrt{1 + \frac{1}{R^2}\Big(\frac{\partial R}{\partial \theta}\Big)^2 + \frac{1}{R^2 \sin^2\theta}\Big(\frac{\partial R}{\partial \varphi}\Big)^2}, \\
\Big(\frac{\partial r}{\partial t} - \frac{\partial R}{\partial \theta}\frac{\partial \theta}{\partial t} - \frac{\partial R}{\partial \varphi}\frac{\partial \varphi}{\partial t}\Big)_2 - \frac{\partial R}{\partial t} = \Big[\Big(\frac{\partial r}{\partial t} - \frac{\partial R}{\partial \theta}\frac{\partial \theta}{\partial t} - \frac{\partial R}{\partial \varphi}\frac{\partial \varphi}{\partial t}\Big)_1 - \frac{\partial R}{\partial t}\Big]\frac{\gamma-1}{\gamma+1}\Lambda, \\
\Lambda \equiv \Big\{1 + \frac{2a_1^2}{\gamma-1}\frac{B^2}{[\frac{\partial R}{\partial t} - (\frac{\partial r}{\partial t} - \frac{\partial R}{\partial \theta}\frac{\partial \theta}{\partial t} - \frac{\partial R}{\partial \phi}\frac{\partial \phi}{\partial t})_1]^2}\Big\} \\
\sin^2\theta \frac{\partial R}{\partial \theta}\Big[R^2\Big(\frac{\partial \theta}{\partial t}\Big)_2 + \frac{\partial R}{\partial \theta}\Big(\frac{\partial r}{\partial t}\Big)_2\Big] + \frac{\partial R}{\partial \varphi}\Big[R^2 \sin^2\theta\Big(\frac{\partial \varphi}{\partial t}\Big)_2 + \frac{\partial R}{\partial \varphi}\Big(\frac{\partial \varphi}{\partial t}\Big)_2\Big] \\
= \sin^2\theta \frac{\partial R}{\partial \theta}\Big[R^2\Big(\frac{\partial \theta}{\partial t}\Big)_1 + \frac{\partial R}{\partial \theta}\Big(\frac{\partial r}{\partial t}\Big)_1\Big] + \frac{\partial R}{\partial \varphi}\Big[R^2 \sin^2\theta\Big(\frac{\partial \varphi}{\partial t}\Big)_1 + \frac{\partial R}{\partial \varphi}\Big(\frac{\partial \varphi}{\partial t}\Big)_1\Big] \\
R^2\Big[1 + \frac{1}{R^2\sin^2\theta}\Big(\frac{\partial R}{\partial \varphi}\Big)^2\Big]\Big(\frac{\partial \theta}{\partial t}\Big)_2 + \frac{\partial R}{\partial \theta}\Big[\Big(\frac{\partial r}{\partial t}\Big)_2 - \frac{\partial R}{\partial \varphi}\Big(\frac{\partial \varphi}{\partial t}\Big)_2\Big] \\
= R^2\Big[1 + \frac{1}{R^2\sin^2\theta}\Big(\frac{\partial R}{\partial \varphi}\Big)^2\Big]\Big(\frac{\partial \theta}{\partial t}\Big)_1 + \frac{\partial R}{\partial \theta}\Big[\Big(\frac{\partial r}{\partial t}\Big)_1 - \frac{\partial R}{\partial \varphi}\Big(\frac{\partial \varphi}{\partial t}\Big)_1\Big] \\
R^2\sin^2\theta\Big[1 + \frac{1}{R^2}\Big(\frac{\partial R}{\partial \theta}\Big)^2\Big]\Big(\frac{\partial \varphi}{\partial t}\Big)_2 + \frac{\partial R}{\partial \varphi}\Big[\Big(\frac{\partial r}{\partial t}\Big)_2 - \frac{\partial R}{\partial \theta}\Big(\frac{\partial \theta}{\partial t}\Big)_2\Big] \\
= R^2\sin^2\theta\Big[1 + \frac{1}{R^2}\Big(\frac{\partial R}{\partial \theta}\Big)^2\Big]\Big(\frac{\partial \varphi}{\partial t}\Big)_1 + \frac{\partial R}{\partial \varphi}\Big[\Big(\frac{\partial r}{\partial t}\Big)_1 - \frac{\partial R}{\partial \theta}\Big(\frac{\partial \theta}{\partial t}\Big)_1\Big], \\
[r]_1^2 = 0, \; [\theta]_1^2 = 0, \; [\varphi]_1^2 = 0, \; [m]_1^2 = 0, \; \Big[\hat{\theta}\Big]_1^2 = 0, \; [\hat{\varphi}]_1^2 = 0.
\end{gathered}
\tag{2}
$$

4 Exact solution of a problem on rotation of a gravitating homogeneous gas ellipsoid of Jacobi (pre-shock wave solution). As initial data, consider the exact solution to the problem of stationary solid rotation of a homogeneous three-axis Jacobi gas ellipsoid.

In spherical coordinates, the exact solution (1) gives the distribution of velocity, pressure and gravitational potential within a three-axis homogeneous ellipsoid of Jacobi [9]

$$
\begin{gathered}
u_r = u_\theta = 0,\ u_\varphi = r\sin\theta\tfrac{d\varphi}{dt},\ \tfrac{d\varphi}{dt} = \omega = const,\ \rho = \rho_0 = const,\\
p(r,\theta,\varphi) = \tfrac{\rho_0}{2}[\omega^2\sin^2\theta - 2(P\sin^2\theta\cos^2\varphi + Q\sin^2\theta\sin^2\varphi + Rcos^2\theta)]A,\\
A \equiv r^2 - \frac{a^2}{\sin^2\theta(1+I_1^2\sin^2\varphi)+(I_2^2+1)\cos^2\theta},\\
\Phi(r,\theta,\varphi) = \Phi_0 - Pr^2\sin^2\theta\cos^2\varphi - Qr^2\sin^2\theta\sin^2\varphi - Rr^2\cos^2\theta,\\
\Phi_0 = \pi k\rho_0 abc\int_0^\infty \tfrac{ds}{\Gamma},\ P = \pi k\rho_0 abc\int_0^\infty \tfrac{1}{a^2+s}\tfrac{ds}{\Gamma},\ Q = \pi k\rho_0 abc\int_0^\infty \tfrac{1}{b^2+s}\tfrac{ds}{\Gamma},\\
R = \pi k\rho_0 abc\int_0^\infty \tfrac{1}{c^2+s}\tfrac{ds}{\Gamma},\ \Gamma(s) = \sqrt{(a^2+s)(b^2+s)(c^2+s)},
\end{gathered}
\tag{3}
$$

a, b, c is the half-axis of the three-axis ellipsoid of rotation, wherein the angular speed of rotation satisfies the relation (in the theory of ellipsoidal figures of equilibrium is called the Jacobi formula, [8])

$$
\begin{gathered}
\frac{Q}{I_1^2+1} < P < Q < R < \frac{I_2^2+1}{I_1^2+1}Q,\ a > b > c,\\
\frac{1}{c^2} > \frac{1}{a^2} + \frac{1}{b^2},\ I_1^2 = \frac{a^2-b^2}{b^2},\ I_2^2 = \frac{a^2-c^2}{c^2}.
\end{gathered}
\tag{4}
$$

5 Asymptotic solution behind the shock wave. A detailed qualitative analysis of the equations of motion (1) and boundary conditions (2) shows that the solution in the vicinity beyond the shock wave can be sought in the form of a singular asymptotic decomposition by a small parameter

$$
\begin{gathered}
r(m,\hat\theta,\hat\varphi,\tau) = R_0(\tau) + \varepsilon H(m,\hat\theta,\hat\varphi,\tau) + \underline{O}(\varepsilon^2),\\
R(\theta,\varphi,\tau) = R_0(\tau) + \varepsilon R_1(\theta,\varphi,\tau) + \underline{O}(\varepsilon^2),\\
M(\theta,\varphi,\tau) = M_0(\tau) + \varepsilon M_1(\theta,\varphi,\tau) + \underline{O}(\varepsilon^2),\\
\theta(m,\hat\theta,\hat\varphi,\tau) = \hat\theta + \varepsilon\theta_1(m,\hat\theta,\hat\varphi,\tau) + \underline{O}(\varepsilon^2),\\
\varphi(m,\hat\theta,\hat\varphi,\tau) = \hat\varphi + \sqrt{\varepsilon}\varphi_1(m,\hat\theta,\hat\varphi,\tau) + \underline{O}(\varepsilon^{3/2}),\\
p(m,\hat\theta,\hat\varphi,\tau) = \tfrac{p_0(m,\tau)}{\varepsilon} + p_1(m,\hat\theta,\hat\varphi,\tau) + \underline{O}(\varepsilon),\\
\rho(m,\hat\theta,\hat\varphi,\tau) = \tfrac{p_0(m,\hat\theta,\hat\varphi,\tau)}{\varepsilon} + \rho_1(m,\hat\theta,\hat\varphi,\tau) + \underline{O}(\varepsilon),
\end{gathered}
\tag{5}
$$

where ε is a small parameter $\varepsilon = \frac{\gamma-1}{\gamma+1},\ \tau = \frac{t}{\sqrt{\varepsilon}},\ E = \frac{E_0}{\varepsilon^2},\ E_0 = \underline{O}(1)$.

Substituting the asymptotic decomposition (5) into the system of equations (1) and boundary conditions (2), given also the integral energy equation, for zero (main approximation) approximations of the unknown functions, we obtain

$$
\begin{gathered}
p_0(m,\tau) = R_0'^2(\tau) + \frac{R_0''(\tau)(M_0(\tau)-m)}{4\pi R_0^2(\tau)},\\
M_0(\tau) = \frac{4}{3}\pi R_0^3(\tau),\\
R_0(\tau) = (\frac{75}{4\pi}E_0)^{1/5}\tau^{2/5},\\
\rho_0(m,\hat{\theta},\hat{\varphi},\tau) = p_0^{1/\gamma}(m,\tau)[R_0'^2(T_0(m))]^{-1/\gamma}[1 + \frac{a_1^2(m,\hat{\theta},\hat{\varphi})}{\gamma R_0'^2(T_0(m))}]^{-1},\\
a_1^2(m,\hat{\theta},\hat{\varphi}) = \frac{\gamma p}{\rho_0} = \frac{\gamma A(m,\hat{\theta},\hat{\varphi})}{2}\\
\times[\omega^2\sin^2\hat{\theta} - 2(P\sin^2\hat{\theta}\cos^2\hat{\varphi} + Q\sin^2\hat{\theta}\sin^2\hat{\varphi} + R\cos^2\hat{\theta})],\\
T_0(m) = (\frac{3}{4\pi})^{5/6}(\frac{4\pi}{75E_0})^{1/2}m^{5/6},\\
A(m,\hat{\theta},\hat{\varphi}) = (\frac{3m}{4\pi})^{2/3} - \frac{a^2}{\sin^2\hat{\theta}(1+I_1^2\sin^2\hat{\varphi})+(I_2^2+1)\cos^2\hat{\theta}}.
\end{gathered}
\tag{6}
$$

6 Conclusions. Thus, using the previously proposed asymptotic method of a small parameter (thin shock layer) for a gravitational gas, we found the main approximations for the law of motion and thermodynamic characteristics of the medium behind the shock wave in the problem of a central explosion in a stationary solid-state rotating with a constant angular velocity of a three-axis Jacobi ellipsoid.

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