Reports of Enlarged Sessions of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 36, 2022

DARBOUX TYPE PROBLEM FOR ONE NONLINEAR HYPERBOLIC EQUATION OF THE FOURTH ORDER

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Abstract. Darboux type problem for one nonlinear hyperbolic equation of the fourth order is considered. The theorem on existence, uniqueness and nonexistence of solutions of this problem are proved.

Keywords and phrases: Nonlinear fourth order hyperbolic equations, Darboux type problem, existence, uniqueness and nonexistence of solutions.

AMS subject classification (2010): 35G30.

On a plane of variables x and t consider the following fourth order hyperbolic equation

$$\Box^2 u + f(\Box u) + g(u) = F(x,t), \tag{1}$$

where $\Box := \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$, f, g and F are given, while u is an unknown scalar function.

Denote by $D_T : 0 < x < t,\, t < T$ the angular domain bounded by characteristic segment $\gamma_{1,T} : x = t,\, 0 \le t \le T, \gamma_{2,T} : x = 0,\, 0 \le t \le T$, and $\gamma_{3,T} : t = T,\, 0 \le x \le T$, temporal and spatial orientation segments, respectively.

For equation (1) in the domain D_T consider the Darboux type boundary value problem with the following statement: find in the domain D_T a solution $u = u(x,t)$ to equation (1) which satisfies on the parts of the boundary $\gamma_{1,T}$ and $\gamma_{2,T}$ the following conditions

$$u|_{\gamma_{1,T}} = u(t,t) = \mu_1(t),\ \frac{\partial u}{\partial \nu}|_{\gamma_{1,T}} = \frac{\partial u}{\partial \nu}(t,t) = \mu_2(t),\ \ 0 \le t \le T, \tag{2}$$

$$u|_{\gamma_{2,T}} = u(0,t) = \mu_3(t),\ \frac{\partial^2 u}{\partial x^2}|_{\gamma_{2,T}} = \frac{\partial^2 u}{\partial x^2}(0,t) = \mu_4(t),\ 0 \le t \le T, \tag{3}$$

where $\mu_i, i = 1, ..., 4$, are given functions, and functions μ_1, μ_3 satisfy on the common point $O = O(0,0)$ of curves $\gamma_{1,T}$ and $\gamma_{2,T}$ the agreement condition $\mu_1(0) = \mu_3(0)$, $\nu = (\nu_x, \nu_t)$ is a unit vector of the outer normal to the boundary ∂D_T.

It should be noted that in the case of a plane or two independent variables, in the theory of differential equations and systems of hyperbolic type and its applications, along with mixed problems, the boundary value problems such as Goursat, Darboux, and other problems of this type are no less important. In some ways these problems can be thought of as a limit case for mixed problems when the initial data carrier converges at a single point. These problems describe the process of gas absorption by the sorbent, the harmonic oscillation of a wedge in the supersonic flow, the oscillation of a wire in the viscous liquid, and so on.

For the following kind of second-order hyperbolic equation

$$u_{tt} - u_{xx} + a(x,t)u_x + b(x,t)u_t + f(x,t,u) = F(x,t)$$

in the linear case (i.e., when the function $f(x,t,u)$ is linear with respect to u), the Goursat and Darboux type problems in the D_T angular domain, when on the part of the boundary $\gamma_{1,T}$ and $\gamma_{2,T}$ are given the Dirichlet or Neuman conditions are well studied (see, for example, works [1] - [10] and others) and their correctness is proved. In the nonlinear case, however, the study of these problems additionally faces substantial difficulties. In some cases, new effects occur, especially when the power nonlinearity in the equation with respect to the sought solution is greater than one. The novelty that can arise in the nonlinear case consists of the violation of global solvability. For example, when

$$f(x,t,u) \leq -C|u|^{\alpha_0} \ \forall u \in \mathbb{R},$$

where $C = const > 0$, $\alpha_0 = const > 1$, for a sufficiently wide class of functions $F(x,t)$ in the equation the corresponding problem does not have a solution in the angular domain D_T. However, when the power nonlinearity is less than one, the global solvability occurs, but the uniqueness of the solution may be violated. The examples are constructed when the problem has a continuum set of linearly independent solutions. For example, the Darboux problem:

$$u_{tt} - u_{xx} - |u|^{\alpha} = 0, \ (x,t) \in D_T,$$
$$u|_{\gamma_{1,T}} = u(t,t) = 0, \ 0 \leq t \leq T,$$
$$u_x|_{\gamma_{2,T}} = u_x(0,t) = 0, \ 0 \leq t \leq T,$$

when $\frac{1}{2} < \alpha < 1$ has an infinite number of linearly independent solutions depending on the parameter $\sigma \geq 0$:

$$u_\sigma(x,t) = \begin{cases} \beta[(t-\sigma)^2 - x^2]^{\frac{1}{(1-\alpha)}}, \ t > \sigma + |x|, \\ 0, \ |x| \leq t \leq \sigma + |x|, \end{cases}$$

where $\beta = [2(1-\alpha)]^{-\frac{2}{(1-\alpha)}}$. These questions have been studied in the works of S. Kharibegashvili, O. Jokhadze, B. Midodashvili, G. Berikelashvili and others.

It should be noted that equations of type (1) are obtained from second-order hyperbolic systems by the method of excluding unknowns.

Remark 1. Let $f, g \in C(\mathbb{R})$, and $F \in C(\bar{D}_T)$. If u, where $u, \Box u \in C^2(\bar{D}_T)$ represents a classical solution of problem (1)-(3), then by function $v = \Box u$ this problem with respect to unknown functions u and v can be reduced to the following boundary value problem

$$L_1(u,v) := \Box u - v = 0, \ (x,t) \in D_T, \tag{4}$$

$$L_2(u,v) := \Box v + f(v) + g(u) = F(x,t), \ (x,t) \in D_T, \tag{5}$$

$$u|_{\gamma_{1,T}} = u(t,t) = \mu_1(t), \ u|_{\gamma_{2,T}} = u(0,t) = \mu_3(t), \ 0 \leq t \leq T, \tag{6}$$

$$v|_{\gamma_{1,T}}=v(t,t)=-\sqrt{2}\mu_2'(t),\ v|_{\gamma_{2,T}}=v(0,t)=\mu_3''(t)-\mu_4(t),\ 0\le t\le T. \quad (7)$$

Here in obtaining the first equality of (7) we took into account that $\frac{d}{dt}w(t,t)=\big(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\big)w|_{x=t}$, $\frac{\partial}{\partial\nu}|_{\gamma_{1,T}}=\frac{1}{\sqrt{2}}(\frac{\partial}{\partial x}-\frac{\partial}{\partial t})$ and, therefore,

$$v|_{\gamma_{1,T}}=\Box u|_{\gamma_{1,T}}=\Big(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\Big)\Big(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\Big)u|_{\gamma_{1,T}}=-\sqrt{2}\Big(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\Big)\frac{\partial u}{\partial\nu}|_{\gamma_{1,T}}=-\sqrt{2}\mu_2'(t),$$

while in obtaining the second equality of (7) we took into account (2),(3), $v=\Box u=\frac{\partial^2u}{\partial t^2}-\frac{\partial^2u}{\partial x^2}$ and, therefore, $v|_{\gamma_{2,T}}=v(0,t)=\Big(\frac{\partial^2u}{\partial t^2}\Big)(0,t)-\Big(\frac{\partial^2u}{\partial x^2}\Big)(0,t)=\mu_3''(t)-\mu_4(t)$. Vice versa, if $u,v\in C^2(\bar D_T)$ represents a classical solution of problem (4)-(7), where $\mu_1,\mu_4\in C([0,T]),\mu_2\in C^1([0,T]),\ \mu_3\in C^2([0,T])$, then function u is a classical solution of problem (1)-(3).

Definition. Let $f,g\in C(\mathbb{R})$, $F\in C(\bar D_T)$ and for simplicity $\mu_i=0$, $i=1,...,4$. The system of functions u and v is called a generalized solution of the class C to problem (4)-(7) if $u,v\in C(\bar D_T)$ and there exist sequences $u_n,v_n\in \overset{\circ}{C}{}^2(\bar D_T):=\{w\in C^2(\bar D_T):w|_{\gamma_{i,T}}=0,\ i=1,2\}$ such that $\lim_{n\to\infty}||u_n-u||_{C(\bar D_T)}=0$, $\lim_{n\to\infty}||v_n-v||_{C(\bar D_T)}=0$, $\lim_{n\to\infty}||L_1(u_n,v_n)||_{C(\bar D_T)}=0$, $||L_2(u_n,v_n)-F||_{C(\bar D_T)}=0$.

Remark 2. It is clear that the classical solution $u,v\in \overset{\circ}{C}{}^2(\bar D_T)$ of problem (4)-(7) is a generalized solution of the class C of this problem. Under certain conditions, the existence of a generalized solution of the class C of problem (4)-(7) is first proved and then it can be shown that this solution is classical.

Below we assume that

$$f(u)=\lambda e^u,\ g(u)=\mu sinu,\ \lambda,\mu=const. \quad (8)$$

The following lemmas are valid.

Lemma 1. *Let $F\in C(\bar D_T)$ and $\mu_i=0,\ i=1,...,4$. Then for any generalized solution u,v of the class C of problem (4)-(7) the following inequality*

$$|u(x,t)|\le te^t||v||_{L_2(D_t)},\ (x,t)\in D_T,$$

is valid.

Lemma 2. *Let functions f and g satisfy the conditions (8), and $\lambda=const\ge 0$ and $F\in C(\bar D_T)$, $\mu_i=0,\ i=1,...,4$. Then for any generalized solution u,v of the class C of problem (4)-(7) the following estimates are valid*

$$|u(x,t)|\le C_1||F||_{L_2(D_t)}+C_2,\ (x,t)\in D_T,$$

$$|v(x,t)|\le C_3||F||_{L_2(D_t)}+C_4,\ (x,t)\in D_T,$$

where the values $C_i=C_i(t)\ge 0,\ i=1,...,4$, do not depend on functions u,v and F.

The problem (4)-(7) can be equivalently reduced to the system of Volterra type nonlinear integral equations with respect to the functions u and v in the class of continuous functions $C(\bar{D}_T)$, whence, based on the abovementioned lemmas and the Leray-Schauder theorem, the following theorem can be proved.

Theorem 1. *Let the condition (8) be fulfilled, where* $\lambda \geq 0$ *and* $F \in C^1(\bar{D}_T),\ \mu_i = 0,\ i = 1, ..., 4$. *Then problem (4)-(7) has a unique generalized solution of the class* C *which also represents a classical solution of this problem in the domain* D_T.

Remark 3. Note that in the general case, if we do not apply some additional conditions other than smoothness to the functions f, g in equation (1), then problem (4)-(7) may no longer be solvable. Indeed, the following theorem is valid.

Theorem 2. *Let* $f = 0, g \in C^1(\mathbb{R}),\ F_0 \in C^1(\bar{D}_T),\ F_0|_{D_T} > 0$ *and* $F = \beta F_0,\ \beta = const > 0,\ \mu_i = 0,\ i = 1, ..., 4$. *Then, if* $g(u) \leq -|u|^\alpha,\ \alpha = const > 1$ *there exists the number* $\beta_0 = \beta_0(F_0, \alpha) > 0$, *such that when* $\beta > \beta_0$ *problem (4)-(7) does not have a generalized solution of the class* C *in the domain* D_T.

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Received 19.05.2022; revised 28.07.2022; accepted 20.09.2022.

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