Reports of Enlarged Sessions of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 36, 2022

LEBESGUE'S TEST FOR GENERAL DIRICHLET'S INTEGRALS

Nika Areshidze

Abstract. It is a well-known Lebesgue ([1], [4]) test for trigonometric Fourier series. Taberski ([2],[3]) considered real-valued Lebesgue locally integrable functions f, such that

$$\lim_{T\to\infty}\frac{1}{T}\int_T^{T+c}|f(t)|\,dt\ =0;\quad \lim_{T\to\infty}\frac{1}{T}\int_{-T-c}^{-T}|f(t)|\,dt\ =0$$

for every fixed $c>0$. For this class of functions, he defined generalized Dirichlet's integrals. Besides, Taberski ([2], [3]) investigated problems of convergence and (C,1)-summability of these integrals. In this paper the analogue of the Lebesgue test for the generalized Dirichlet's integrals is proved.

Keywords and phrases: Lebesgue's test, general Dirichlet's integrals, convergence.

AMS subject classification (2010): 42A20, 42A38.

One of the most important tests for the convergence of Fourier series are those of Dini, Dini-Lipschitz, and Dirichlet-Jordan, each of which is based on a different idea. In 1905 Lebesgue proved the theorem which is known as Lebesgue's test and which is more general than the others. In 1973 Taberski [3] considered class E of real-valued, Lebesgue locally integrable functions. Taberski ([2], [3]) investigated problems of convergence and (C,1)-summability of these integrals.

Definition. Let E be the class of all real-valued, Lebesgue locally integrable functions f such that

$$\lim_{T\to\infty}\frac{1}{T}\int_T^{T+c}|f(t)|\,dt\ =0;\quad \lim_{T\to\infty}\frac{1}{T}\int_{-T-c}^{-T}|f(t)|\,dt\ =0 \tag{1}$$

for every fixed $c>0$.

Remark 1. If real-valued, Lebesgue locally integrable function f is periodic (with a least positive period m) then (1) is fulfilled. Indeed, there exists $k>0$, such that $c<k\cdot m$. Then we have

$$\frac{1}{T}\int_T^{T+c}|f(t)|\,dt\ \le\frac{1}{T}\int_T^{T+k\cdot m}|f(t)|\,dt$$

$$=\frac{1}{T}\cdot k\int_T^{T+m}|f(t)|\,dt\ =\frac{1}{T}\cdot k\cdot M\to 0$$

when $T\to+\infty$. M is the integral from $|f|$ on the interval $[T;T+m]$. Similarly we show that the second condition in (1) is fulfilled.

Let for any given function $f \in E$ and a positive number l

$$a_k^l = \frac{1}{l}\int_{-l}^{l} f(t)\cos\frac{\pi kt}{l}\,dt\ , \quad b_k^l = \frac{1}{l}\int_{-l}^{l} f(t)\sin\frac{\pi kt}{l}\,dt,$$

$$S_n^l(x;f) = \frac{a_0}{2} + \sum_{k=1}^{n}\left(a_k^l\cos\frac{\pi kx}{l} + b_k^l\sin\frac{\pi kx}{l}\right),$$

where $x \in (-\infty, +\infty)$, $\quad n = 0, 1, 2....$

Let

$$\phi_x(t) = (f(x+t) + f(x-t) - 2f(x)), \quad D_n^l(t) = \frac{\sin\frac{(2n+1)\pi t}{2l}}{2\sin\frac{\pi t}{2l}}.$$

Taberski [3] showed that if $f \in E$ then for every fixed point of $[a;b]$

$$S_n^l(x;f) - f(x) = \frac{1}{l}\int_0^l \phi_x(t)D_n^l(t)\,dt\ + o(1), \quad l \to +\infty, \tag{2}$$

(2) is uniformly in $x \in [a;b]$ $\quad (-\infty < a \le b < +\infty), n = 0, 1, 2, ...,$ if $f \in E$ is bounded on $[a;b]$. It is easy to see that

$$\frac{1}{l}\int_0^l \phi_x(t)D_n^l(t)\,dt = \frac{1}{l}\int_0^l \chi_l(t)\sin\frac{\pi nt}{l}\,dt\ + \frac{1}{2l}\int_0^l \phi_x(t)\cos\frac{\pi nt}{l}\,dt$$

$$= M_n^l(x) + N_n^l(x)$$

where $\chi_l(t) = \frac{1}{2}\phi_x(t)\cot\frac{\pi t}{2l}$. We have

$$\left|\frac{1}{2}\cot\frac{\pi t}{2l}\cdot\sin\frac{\pi nt}{l}\right| \le n\pi, \tag{3}$$

$$\left|\chi_l(t)\sin\frac{\pi nt}{l}\right| \le |\phi_x(t)|, \quad t \in [l-\eta; l]. \tag{4}$$

Let for $x \in [a;b]$

$$\Phi(h) = \int_0^h |\phi_x(t)|\,dt = o(h), \quad h \to +\infty. \tag{5}$$

Using (3)-(5), we have

$$\left|\frac{1}{l}\int_0^l \phi_x(t)D_n^l(t)\,dt\right| \le \frac{\pi}{2}\int_\eta^l \frac{|\phi_x(t) - \phi_x(t+\eta)|}{t}\,dt\ + \eta\int_\eta^l \frac{|\phi_x(t)|}{t^2}\,dt$$

$$+\pi\eta^{-1}\int_0^{2\eta}|\phi_x(t)|\,dt\ +o(1),\quad l\to+\infty. \tag{6}$$

If $f\in E$ is bounded on $[a;b]$ and (5) is satisfied uniformly in $x\in[a;b]$ then (6) is fulfilled uniformly in $x\in[a;b]$.

Theorem. *Suppose $f\in E$ and for $x_0\in[a;b]$*

$$\Phi(h)=\int_0^h|\phi_{x_0}(t)|dt=o(h),\, when\quad h\to0,\quad h\to+\infty, \tag{7}$$

and

$$\int_\eta^l\frac{|\phi_{x_0}(t)-\phi_{x_0}(t+\eta)|}{t}dt\to0,\quad when\quad \eta=\frac{l}{n}\to0,\quad n,l\to+\infty. \tag{8}$$

Then $S_n^l(x;f)$ converges to $f(x_0)$, when $\eta\to0,\quad n\to+\infty,\quad l\to+\infty$. Convergence is uniform on $[a;b]$ if $f\in E$ is bounded and conditions (7) and (8) are satisfied uniformly.

Proof. We apply (6). The first term of (6) is o(1) by hypothesis. The third term there is $\pi\eta^{-1}\Phi(2\eta)=o(1)$ when $\eta\to0$. Integration by parts of the second term gives

$$\eta\int_\eta^l\frac{|\phi_{x_0}(t)|}{t^2}\,dt\ =\frac{\Phi(l)}{n\cdot l}-\frac{\Phi(\eta)}{\eta}+2\eta\int_\eta^l\frac{\Phi(t)}{t^3}\,dt$$

i.e

$$\eta\int_\eta^l\frac{|\phi_{x_0}(t)|}{t^2}\,dt\ =\frac{\Phi(l)}{n\cdot l}-\frac{\Phi(\eta)}{\eta}+2\eta\int_\eta^1\frac{\Phi(t)}{t^3}\,dt\ +2\eta\int_1^l\frac{\Phi(t)}{t^3}\,dt$$

$$=K_1-K_2+K_3+K_4.$$

Obviously, $K_1=o(1),K_2=o(1),$ when $\quad \frac{l}{n}\to0,\quad n\to+\infty,\quad l\to+\infty$.

From (7) we have $\Phi(t)=o(t)$, when $t\to0$. Therefore, for $\forall\varepsilon>0$ there exists $\delta>0$ such that, when $0<t\le\delta$, then $\Phi(t)\le\varepsilon\cdot t$. So we have

$$K_3=2\eta\int_\eta^\delta\frac{\Phi(t)}{t^3}\,dt\ +2\eta\int_\delta^1\frac{\Phi(t)}{t^3}\,dt$$

$$\le2\eta\varepsilon\left(\frac{1}{\eta}-\frac{1}{\delta}\right)+\frac{2\eta}{\delta^3}\int_\delta^1\Phi(t)\,dt\ \le2\varepsilon+\frac{2\eta}{\delta^3}\int_\delta^1\Phi(t)\,dt.$$

Since $\eta\to0$, when $n,l\to+\infty$, therefore, there exists N such that for $n,l\ge N$

$$\eta\le\frac{\varepsilon\cdot\delta^3}{2\int_\delta^1\Phi(t)\,dt}.$$

Whence

$$K_3 \leq 3\varepsilon,$$

if n and l are large enough. Since ε is arbitrary, we get $K_3 = o(1)$, when $n, l \to +\infty$. Besides, from (7) we have $\Phi(t) = o(t)$, when $t \to +\infty$. Therefore, for $\forall\varepsilon > 0$ there exists $s > 1$ such that when $t \geq s$ then $\Phi(t) \leq \varepsilon \cdot t$. Whence

$$K_4 = 2\eta \int_1^s \frac{\Phi(t)}{t^3}\, dt \ + 2\eta \int_s^l \frac{\Phi(t)}{t^3}\, dt$$

$$\leq 2\eta \int_1^s \Phi(t)\, dt \ + 2\eta\varepsilon \left(\frac{1}{s} - \frac{1}{l}\right) \leq 2\varepsilon$$

when n and l are large enough. Therefore, $K_4 = o(1)$ when $n, l \to +\infty$. The first part of the theorem is proved. The second part of the theorem will be similarly proved. □

Remark 2. In particular, if a function $f \in E$ is continuous on $(a'; b')$, then the first condition of (7) is satisfied uniformly over any closed interval $[a; b]$ $(a' < a \leq b < b')$.

REFERENCES

1. Lebesgue H. Recherches sur la convergence des séries de Fourier. *Math. Ann.*, **61** (1905), 251-280.
2. Taberski, R. Convergence of some trigonometric sums. *Demonstratio Mathematica*, **5** (1973), 101-117.
3. Taberski, R. On general Dirichlet's integrals. *Anales soc. Math Polonae, Series I: Prace mathematyczne*, **XVII** (1974), 499-512.
4. Zygmund, A. Trigonometric Series. *Cambridge University Press*, **1** (1959).

Received 11.05.2022; accepted 29.08.2022.

Author(s) address(es):

Nika Areshidze
Department of Mathematics, I. Javakhishvili Tbilisi State University
University str. 2, 0186 Tbilisi, Georgia
E-mail: nika.areshidze804@ens.tsu.edu.ge, nika.areshidze15@gmail.com