ON IRREDUCIBLE ALGEBRAIC SETS OVER FREE 2-NILPOTENT GROUPS *

Mikheil Amaglobeli   Alexei Miasnikov

Abstract. In this paper we given a pure algebraic description of the coordinate groups of irreducible algebraic sets over non-abelian free 2-nilpotent group N.

The basic concepts of algebraic geometry over groups were described in the works by Baumslag, Myasnikov, and Remeslennikov [1], which we follow in the terminology and notation. More general, universal algebraic geometry applicable to arbitrary algebraic systems was initiated by Plotkin, Daniyarova, Miasnikov, and Remeslennikov and has been successfully developed for semigroups, noncommutative rings, semilattices, and graphs. At present, algebraic geometry over groups has become an important research tool in combinatorial geometric, and model-theoretic group theory. The most thoroughly developed are algebraic-geometric methods for free groups, hyperbolic groups, and partially commutative groups, as well as for metabelian, free solvable, and rigid solvable groups. An open question of fundamental importance in this area is the construction of algebraic geometry over torsion-free nilpotent groups, in particular, over free nilpotent groups. In addition to direct interest in nilpotent groups, the importance of this issue lies in that the Fitting subgroup Φ of a group G is often distinguished from G by a finite system of equations, so algebraic geometry over the nilpotent group Φ is directly embedded in algebraic geometry over the initial possibly non-nilpotent group G.

In this paper, we describe the algebraic properties of coordinate groups of algebraic sets and their irreducible components over a group N (the Zariski topology over N is Noetherian, so every algebraic set is the finite union of its irreducible components). Due to these results, we can hope that other fundamental problems in algebraic geometry over N would be solved, for example, a reasonable description of solution sets of finite systems of equations over N would be obtained (despite the Diophantine problem over N is unsolvable).

Note that, in algebraic geometry over an arbitrary group N, it is natural to consider groups containing N as a subgroup (so-called N-groups) and homomorphisms of N-groups that are identical on N (N-homomorphisms). Our approach to the description

---

of coordinate groups is based on some discrimination results, which are also of interest in themselves. Recall that an N-group H is N-separated (N-discriminated) by an N-group G if for any $h \in H$, $h \neq 1$ (for any nontrivial $h_1, \ldots, h_n \in H$) there exists an N-homomorphism $\varphi : H \to G$ such that $\varphi(h) \neq 1$ ($\varphi(h_i) \neq 1$, $i = 1, \ldots, n$). If $N = 1$, then we obtain the standard concepts of separation and discrimination, which appear in various areas of group theory.

In 1967 Baumslag [2] proved that a group H is discriminated by a non-Abelian free group F if and only if it is separated by F and is commutative transitive, or a CT-group (the commutative relation is transitive on the set of nontrivial elements of H, i.e., the centralizers of nontrivial elements of H are Abelian). Later, similar results were found to hold for many other groups (e.g., for torsion-free hyperbolic groups). However, non-Abelian nilpotent groups are never CT-groups, since they always have a nontrivial center. Nevertheless, it turned out that the definition of a CT-group an be slightly generalized so that the new definition works in classes of nilpotent groups as well. Namely, a group H is called a CT-group of level k, $k = 0, 1, \ldots$, or a CT_k-group it the centralizer of any element that is not in $Z_k(H)$ is Abelian; here, $Z_k(H)$ is the k-th term in the upper central series of H. Specifically, if $k = 0$, then $Z_0(H) = 1$, so CT_0-groups are exactly CT-groups. Note that, for CT_1-groups, the centralizers of noncentral elements are Abelian. Clearly, N is a CT_1-group. The concept of a CT_1-group we introduced [3].

Proposition. *Let N be a non-Abelian free k-nilpotent group of finite rank. If an N-group H is N-discriminated by N, then H is a CT_{k-1}-group.*

In particular, if N is a 2-nilpotent group, then an N-group H that is N-discriminated by N is a CT_1-group.

Now, we describe our main results on algebraic geometry. For this purpose, we need some definitions.

Let G be a group from the variety $\mathcal{N}_2$ of nilpotent groups of step ≤ 2. The Cartesian power $G^n = G \times \cdots \times G$ (n copies) is called the affine space over G. Let $X = \{x_1, \ldots, x_n\}$ be a set of variables and let

$$G[X] = G \underset{\mathcal{N}_2}{*} F(X)$$

be the nilpotent product, where $F(X)$ is a free nilpotent group in $\mathcal{N}_2$ with base X. A system of equations S (or $S > 1$) over G is an arbitrary subset of $G[X]$. An element $u \in S$ can be treated as a group word in variables $x_1, \ldots, x_n$ with coefficients from G, $u = u(x_1, \ldots, x_n)$. An element $p = (g_1, \ldots, g_n) \in G^n$ is called a solution of the system $S = 1$ in G if $u(g_1, \ldots, g_n) = 1$ in G for every $u \in S$. A subset V of the affine space G^n is called a algebraic set over G if V is the set of all solutions of a system of equation S from $G[X]$; in this case, we write $V = V_G(S)$. Additionally, for $V = V_G(S)$, we define

$$Rad(V) = \big\{ u \in G[X] \mid u(p) = 1 \, \forall \, p \in V_G(S) \big\}.$$

Obviously, $Rad(V)$ always a normal subgroup in $G[X]$. The group $\Gamma(V) = G[X]/Rad(V)$ is called a coordinate group of the algebraic set V. Note that $\Gamma(V)$ is a G-group with respect to the natural embedding $G \to \Gamma(V)$.

Using, as a prebasis of closed sets, all algebraic sets of G^n, we turn G^n into a topological space (Zariski topology). An irreducible algebraic set in G^n is defined in the standard manner. It is well known that, for any linear group G, specifically, for every finitely generated nilpotent group, the Zariski topology on G^n is Noetherian; therefore, each algebraic set in G^n is a finite union of irreducible algebraic sets.

Theorem 1. *Let N be a non-Abelian free nilpotent group of finite rank and let H be a finitely generated N-group. Then the following conditions are equivalent:*

(1) *H is a coordinate group of an algebraic set in N^n for some positive integer k.*

(2) *H is N-separated by N.*

(3) *H is an N-subgroup of a finite direct product $N^k = N_1 \times \cdots \times N_k$ of groups $N_i \cong N$ for some positive integer k in which N is diagonally embedded.*

Finally, we can state the main result.

Theorem 2. *Let N be a non-Abelian free 2-nilpotent group of finite rank and let H be a coordinate group of an irreducible algebraic set from N^n. Then the following conditions are satisfied:*

(1) *H is N-discriminated by N.*

(2) *H is an N-subgroup of finite direct product $N^k = N_1 \times \cdots \times N_k$ of groups $N_i \cong N$ for some positive integer k in which N is diagonally embedded; moreover, H is a CT_1-group.*

(3) *H is an N-subgroup of a finite direct product $N^k = N_1 \times \cdots \times N_k$ of groups $N_i \cong N$ for some positive integer k is which N is diagonally embedded; moreover, for any $i = 1, \ldots, k$, the intersection $H \cap N_i$ is an Abelian normal subgroup in H.*

Conjecture. Conditions (1), (2) and (3) for the groups N and H in Theorem 2 are equivalent.

REFERENCES

1. BAUMSLAG, G., MYASNIKOV, A., REMESLENNIKOV, V. Algebraic geometry over groups. I. Algebraic sets and ideal theory. *J. Algebra*, **219**, 1 (1999), 16-79.

2. BAUMSLAG, B. Residually free groups. *Proc. London Math. Soc. (3)*, **17** (1967), 402-418.

3. LEVIN, F., ROSENBERGER, G. On power-commutative and commutation-transitive groups. *Proceedings of group-St. Andrews 1985*, 249–253, London Math. Soc. Lecture Note Ser., 121, *Cambridge Univ. Press, Cambridge*, 1986.

Author(s) address(es):

Mikheil Amaglobeli
Faculty of Exact and Natural Sciences, I. Javakhishvili Tbilisi State University
University str. 2, Tbilisi 0186, Georgia
E-mail: mikheil.amaglobeli@tsu.ge

Alexei Miasnikov
Stevens Institute of Technology
Hoboken, USA
E-mail: amiasnikov@gmail.com