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THE CONSISTENT ESTIMATORS OF CHARLIER STATISTICAL STRUCTURES

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Abstract. The Charlier statistical structure is determined and the necessary and sufficient conditions for the existence of consistent estimators of the parameters are given.

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1 Introduction. Let (E, S) be a measurable space with a given family of probability measures $\{\mu_i, i \in I\}$. The following definitions are taken from [1]-[3].

Definition 1. An object $\{E, S, \mu_i, i \in I\}$ is called a statistical structure.

Definition 2. A statistical structure $\{E, S, \mu_i, i \in I\}$ is called orthogonal (singular) if $\{\mu_i, i \in I\}$ consists of pairwise singular measures (i.e. $\mu_i \perp \mu_j$, $\forall i \neq j$).

Definition 3. A statistical structure $\{E, S, \mu_i, i \in I\}$ is called weakly separable if there exists a family of S-measurable sets $\{X_i, i \in I\}$ such that

$$\mu_i(X_j) = \begin{cases} 1, & if\ i = j; \\ 0, & if\ i \neq j \end{cases} \qquad (i, j \in I).$$

Definition 4. A statistical structure $\{E, S, \mu_i, i \in I\}$ is called separable if there exists a family of S-measurable sets $\{X_i, i \in I\}$ such that

$$1)\ \mu_i(X_j) = \begin{cases} 1, & if\ i = j; \\ 0, & if\ i \neq j \end{cases} \qquad (i, j \in I);$$

$$2)\ \forall i, j \in I: \ \ card(X_i \cap X_j) < c, \ \ if\ i \neq j,$$

where c denotes the power of continuum.

Definition 5. A statistical structure $\{E, S, \mu_i, i \in I\}$ is called strongly separable if there exists a disjoint family of S-measurable sets $\{X_i, i \in I\}$ such that $\cup_{i\in I} X_i = E$ and $\mu_i(X_i) = 1, \ \forall i \in I$.

Let $B(I)$ be a σ-algebra of subsets of I which contains all finite subsets.

Definition 6. We will say that the statistical structure $\{E, S, \mu_i, i \in I\}$ admits a consistent estimator of parameter $i \in I$ if there exists at least one measurable mapping $\delta : (E, S) \longrightarrow (I, B(I))$, such that

$$\mu_i(\{x : \delta(x) = i\}) = 1, \ \ \forall\, i \in I.$$

Let M^σ be a real linear space of all alternating finite measures on S.

Definition 7. A linear subset $M_B \subset M^\sigma$ is called a Banach space of measures if:

1) a norm on M_B can be defined so that M_B will be a Banach space with respect to this norm, and for any orthogonal measures $\mu, \nu \in M_B$ and real number $\lambda \neq 0$ the inequality $||\mu + \lambda\nu|| \geq ||\mu||$ is fulfilled;

2) if $\mu \in M_B$, $|f(x)| \leq 1$, then $\nu_f(A) = \int_A f(x)\mu(dx) \in M_B$ and $||\nu_f|| \leq ||\nu||$;

3) if $\nu_n \in M_B$, $\nu_n > 0$, $\nu_n(E) < \infty$, $n = 1, 2, ...$ and $\nu_n \downarrow 0$, then for any linear functional $l^* \in M_B^*$: $\lim\limits_{n \longrightarrow \infty} l^*(\nu_n) = 0$, where M_B^* is conjugate to the Banach space M_B.

Definition 8. Let I be some set of indexes and let M_{B_i} be a Banach space for all $i \in I$. We set

$$M_B = \{\{X_i\}_{i \in I} : X_i \in M_{B_i}, \sum_{i \in I} ||X_i||_{M_{B_i}} < \infty\}.$$

Then the M_B with the norm $||\{X_i\}_{i \in I}|| = \sum_{i \in I} ||X_i||_{M_{B_i}}$ is a Banach space. It is called the direct sum of Banach spaces M_{B_i} and is denoted as $M_B = \oplus_{i \in I} M_{B_i}$.

The following theorem is also proved in the paper [3].

Theorem 1. *If M_B is a Banach space of measures, then in M_B there exists a family of pairwise orthogonal probability measures $M = \{\mu_i, i \in I\}$ such that $M_B = \oplus_{i \in I} M_{B_i}$, where M_{B_i} is the Banach space of elements ν such that:*

$$\nu(B) = \int_B f(x)\mu_i(dx), \quad ||\nu||_{M_{B_i}} := \int_E |f(x)|\mu_i(dx) < \infty.$$

2 The consistent estimators of Charlier statistical structures.

Let μ_{Ch} be the probability Charlier measure given on R by the formula

$$\mu_{Ch}(A) = \int_A f_{Ch}(x)dx, \ \ A \in B(R),$$

where $f_{Ch}(x)$ is the Charlier spectral densities. Let $\{\mu_i, i \in I\}$ is a corresponding Charlier probability measures.

Definition 9. The statistical structure $\{E, S, \mu_i, i \in I\}$, where μ_i are the Charlier probability measures, is called the Charlier statistical structure.

Let $\{E, S, \mu_i, i \in I\}$ be the Charlier statistical structure. Consider S-measurable functions $g_i(x)$ $(i \in I)$ such that

$$\sum_{i \in I} \int_E |g_i|(x)\mu_i(dx) < \infty.$$

Let M_B be the set of measures defined by formula

$$\nu(B) = \sum_{i \in I_1} \int_B g_i(x)\mu_i(dx) < \infty,$$

where $I_1 \subset I$ is a countable subset of I and

$$\sum_{i \in I_1} \int_E |g_i|(x)\mu_i(dx) < \infty.$$

If we define the norm in M_B by the formula $||\nu|| = \sum_{i\in I} \int_E |g_i|(x)\mu_i(dx)$, then M_B is a Banach space of measures and $M_B = \oplus_{i\in I} M_{B_i}$, where M_{B_i} is the Banach space of elements ν such that (see [3]):

$$\nu(B) = \int_B g_i(x)\mu_i(dx), \quad ||\nu||_{M_{B_i}} := \int_E |g_i(x)|\mu_i(dx) < \infty.$$

Let $M_B = \oplus_{i\in I} M_{B_i}$, $I = \{i_1, i_2, ...\}$, be the Banach space of measures, let E be a complete separable metric space, and let S be the σ-algebra on E. Denote by $F = F(M_B)$ the set of real functions f for which integral $\int_E f(x)\mu_i(dx)$ is defined $\forall \mu_i \in M_B$ $(i \in I)$.

Theorem 2. *In order that the Charlier orthogonal statistical structure $\{E, S, \mu_i, i \in I\}$ admit consistent estimators of parameters it is necessary and sufficient that the correspondence $f \in l_f$ defined by the equality*

$$\int_E f(x)\mu_i(dx) = l_f(\mu_i), \quad \forall \mu_i \in M_B$$

was one-to-one (here l_f is a linear continuous functional on M_B, $f \in F(M_B)$).

Proof. **Necessity.** The existence of a consistent estimator $\delta : (E, S) \longrightarrow (I, B(I))$ of the parameter $i \in I$ implies that $\forall i \in I : \mu_i(\{x : \delta(x) = i\}) = 1$. Setting $X_i = \{x : \delta(x) = i\}$ for $i \in N$ we get: 1) $\mu_i(X_i) = 1$; $X_i \cap X_j = \emptyset$, $i \neq j$. Therefore, the statistical structure $\{E, S, \mu_i, i \in I\}$ is strongly separable, so there exist S-measurable sets $\{X_i\} i \in I$ such that

$$\mu_i(X_j) = \begin{cases} 1, & if\ i = j; \\ 0, & if\ i \neq j. \end{cases}$$

We associate with the function $I_{x_i} \in F(M_B)$ a continuous linear functional by the formula

$$\int_E I_{x_i}(x)\mu_i(dx) = l_{I_{x_i}}(\mu_i) = ||\mu_i||_{M_{B_i}}.$$

Let us put the linear continuous functional $I_{\widetilde{f_1}}$ in correspondence with the function $\widetilde{f_1}(x) = f_1(x) I_{X_i}(x)$. Then for $\mu_{i_1} \in M_B(\mu_i)$ we have

$$\int_E \widetilde{f_1}(x)\mu_{i_1}(dx) = \int_E f_1(x) I_{X_i}(x)\mu_{i_1}(dx)$$

$$= \int_E f(x) f_1(x) I_{X_i}(x) \mu_i(x) dx = l_{\widetilde{f}_1}(\mu_{i_1}) = ||\mu_{i_1}||_{M_{B_i}}.$$

If we put now the linear continuous functional I_f in correspondence with the function $f(x) = \sum_{i \in N} g_i(x) I_{X_i}(x) \in F(M_B)$ then we obtain

$$\int_E f(x) \mu(dx) = ||\mu|| = \sum_{i \in N} ||\mu_i||_{M_{B_i}},$$

where $\mu(B) = \sum_{i \in N} \int_B g_i(x) \mu_i(dx)$, $B \in S$.

Sufficiency. For $f \in F(M_B)$ we define linear continuous functional l_f by the equality $\int_E f(x)\mu(dx) = l_f(\mu)$. Denote by I_f the countable subset in I for which $\int_E f(x)\mu_i(dx) = 0$ for $i \notin I_f$. The corresponding functional on M_{B_i} denote by l_{f_i}. Then for $\mu_{i_1}, \mu_{i_2} \in M_{B_i}$ we have

$$\int_E f_{i_1}(x) \mu_{i_2}(dx) = l_{f_1}(\mu_{i_2}) = \int_E f_1(x) f_2(x) \mu_{i_1}(dx) = \int_E f_{i_1}(x) f_2(x) \mu_{i_1}(dx).$$

Therefore $f_{i_1} = f_1$ a. e. with respect to the measure μ_{i_1}.

Let $f_i > 0$ a. e. with respect to the measure μ_i and $\int_E f_i(x)\mu_i(dx) < \infty$, then $\int_E f_i(x)\mu_j(dx) = l_{f_i}(\mu_j) = 0, \forall j \neq i$. Denote by $C_i = \{x : f_i(x) > 0\}$, then $\int_E f_i(x)\mu_j(dx) = 0$, $\forall j \neq i$. Hence, it follows that $\mu_j(C_i) = 0$, $\forall j \neq i$. On the other hand, $\mu_j(E \setminus C_i) = 0$. Therefore, the statistical structure $\{E, S, \mu_i, i \in I\}$ is weakly separable. Hence, there exists the family of S-measurable sets $X_i, i \in I$ such that

$$\mu_i(X_j) = \begin{cases} 1, & if\ i = j; \\ 0, & if\ i \neq j. \end{cases}$$

Consider now the sets $\overline{X}_i = X_i \setminus (X_i \cap (\cup_{k \neq i} X_k))$, $i \in I$. It is obvious that these sets are S-measurable disjoint sets and $\mu_i(\overline{X}_i) = 1$, $i \in I$. Let us now define the mapping $\delta : (E, S) \longrightarrow (I, B(I))$ in the following way: $\delta(\overline{X}_i) = i$, $i \in I$. Hence, δ is the consistent estimator for parameter $i \in I$. □

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