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ON THE NUMBER OF REPRESENTATIONS OF POSITIVE INTEGERS BY SOME DIAGONAL QUADRATIC FORMS IN NINE VARIABLES WITH COEFFICIENTS THAT ARE ONES AND FOURS

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Abstract. The modular properties of generalized theta-functions with characteristics and spherical functions are used to build cusp forms of half-integral weight. It gives the opportunity of obtaining formulas for the numbers of representations of positive integers by some diagonal quadratic forms with coefficients that are ones and fours.

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1 Introduction.

Let $f = f(x) = f(x_1, x_2, \ldots, x_s) = \frac{1}{2}x'Ax = \frac{1}{2}\sum\limits_{j,k=1}^{s} a_{jk}x_jx_k$ be an integral positive quadratic form. Here x is a column-vector, and x' is a row-vector with components $x_1, x_2, \ldots, x_s$. Let further $r(n; f)$ denote the number of representations of a positive integer n by the form f.

It is well known that the function $r(n; f)$ can be expressed in the form

$$r(n; f) = \rho(n; f) + \nu(n; f),$$

where $\rho(n; f)$ is a "singular series" and the formulae for its calculation are known ([1–3]). The second summand $\nu(n; f)$ is a Fourier coefficient of a cusp form. It can be expressed in terms of modular forms as follows:

$$\begin{gathered} \theta(\tau; f) = E(\tau; f) + X(\tau), \\ \theta(\tau; f) = 1 + \sum_{n=1}^{\infty} r(n; f)Q^n, \end{gathered} \tag{1}$$

where $\tau \in H = \{\tau : \operatorname{Im}\tau > 0\}$. $Q = e^{2\pi i\tau}$, $X(\tau)$ is a cusp form, and

$$E(\tau; f) = 1 + \sum_{n=1}^{\infty} \rho(n; f)Q^n. \tag{2}$$

is an Eiserstein series corresponding to f. If the genus of the quadratic form f contains one class, then according to Siegel's theorem $\theta(\tau; f) = E(\tau; f)$ and therefore the problem of obtaining exact formulas for $r(n; f)$ is solved completely. If the genus contains more than one class, then it is necessary to determine the cusp form $X(\tau)$. Many papers are devoted to the problem of finding $X(\tau)$. The cusp forms in these works are constructed in the form of linear combinations of product of simple theta-functions with characteristics or their derivatives (see, e.g., [2, 3]). All these functions are special cases of linear combinations of the so-called generalized theta-functions with characteristics and spherical function defined below by (3) (see [4]).

In the present paper, using modular properties of these functions, a cusp form weight 9/2 is constructed which belongs to the space of entire modular forms of type $(-9/2.48, \nu(M))$. Here

$$M \in \Gamma_0(48) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(z) | c \equiv 0 \pmod{48} \right\}$$

and $\nu(M)$ is a system of multipliers respect to the function $\theta(\tau, f^{(k)})$, where

$$f^{(k)} = \sum_{j=1}^{k} x_j^2 + 4 \sum_{j=k+1}^{9} x_j^2 \quad (k = 1, 2, \ldots, 8).$$

All of these functions have the same multiplier system.

This cusp form can be used to obtain exact formulas for the number of representations of positive integers by the forms $f^{(k)}$. In this paper the cases $k = 4$ and $k = 3$ are considered.

Definition 1 ([4], p. 438). Let $f = \frac{1}{2} x' A x$ be a positive definite quadratic form, let A be an integral matrix with even diagonal elements, and let $x \in Z^s$, $s \geq 2$, be a column vector. Moreover g and h be special vectors with respect to f, let $p_\nu = p_\nu(x)$ be a spherical function of order ν with respect to A, and let N be the level of f. Then we define the corresponding generalized theta-function with characteristics and spherical function as follows:

$$\theta_{gh}(\tau; P_\nu, f) = \sum_{x \equiv g \pmod{N}} (-1)^{\frac{h' A (x-g)}{N^2}} P_\nu(x) e^{\frac{\pi i \tau x' A x}{N^2}}. \tag{3}$$

2 Basic Results.

Theorem 1. *Let*

$$\begin{gathered} f = x_1^2 + x_2^2 + \cdots + x_4^2 + 4(x_5^2 + \cdots + x_9^2), \\ f_1 = 12x_1^2 + 12x_2^2 + 12x_3^2, \quad (g^{(1)})' = (8, 8, 8), \quad (g^{(2)})' = (16, 16, 16), \\ (g^{(3)})' = (8, 8, 16), \quad (g^{(4)})' = (16, 16, 8), \quad h' = (0, 0, 0). \quad P_3 = x_1 x_2 x_3. \end{gathered}$$

Then the equality

$$\begin{aligned} \theta(\tau; f) = & E(\tau; f) + \frac{1}{68} \theta_{g^{(1)}h}(\tau; P_3, f_1) + \frac{3}{17 \cdot 256} \theta_{g^{(2)}h}(\tau; P_3, f_1) \\ & + \frac{3 \cdot 23}{17 \cdot 256} \theta_{g^{(3)}h}(\tau; P_3, f_1) + \frac{11}{17 \cdot 128} \theta_{g^{(4)}h}(\tau; P_3, f_1) \end{aligned}$$

holds, where the functions

$$\theta(\tau,f),\quad \theta_{g^{(k)}h}(\tau,P_3,f_1)\quad (k=1,2,3,4)$$

are defined by formulas (1) and (3), while the function $E(\tau;f)$ by formula (2).

Theorem 1 can be proved by means of the theory of modular forms. The additional terms are built by the functions (3).

Equating the coefficients of Q^n in both sides of the identity from Theorem 1, we get

Theorem 2. *Let* $f=x_1^2+x_2^2+\cdots+x_4^2+4(x_5^2+\cdots+x_9^2)$, $n=2^\alpha u(2\dagger u)$, $4^5n=r^2\omega$ (ω *is a square free integer*), $r_1^2|u$. *Then*

$$\begin{aligned}
r(n;f)=&\frac{768u^{\frac{7}{2}}}{17\pi^4r_1^7}L(4,\omega)\sum_{d|r_1}d^7\prod_{p|d}\Big(1-\Big(\frac{\omega}{p}\Big)p^{-4}\Big)\\
&+\frac{128}{17}\sum_{\substack{x_1^2+x_2^2+x_3^2=3n\\ x_1\equiv x_2\equiv x_3\equiv 1 \,(\mathrm{mod}\ 6)}}x_1x_2x_3\quad \textit{if}\quad n\equiv 1\ (\mathrm{mod}\ 4),\\
=&\frac{768\cdot 2^{\frac{7\alpha}{2}}u^{\frac{7}{2}}}{17\pi^4r_1^7}\chi_2L(4,\omega)\sum_{d|r_1}d^7\prod_{p|d}\Big(1-\Big(\frac{\omega}{p}\Big)p^{-4}\Big)\\
&+\frac{48}{17}\sum_{\substack{4(x_1^2+x_2^2+x_3^2)=3n\\ x_1\equiv x_2\equiv x_3\equiv 1 \,(\mathrm{mod}\ 3)}}x_1x_2x_3\quad \textit{if}\quad n\equiv 0\ (\mathrm{mod}\ 4),\\
=&\frac{768\cdot n^{\frac{7}{2}}}{17\pi^4r_1^7}\chi_2L(4,\omega)\sum_{d|r_1}d^7\prod_{p|d}\Big(1-\Big(\frac{\omega}{p}\Big)p^{-4}\Big)+\\
&+\frac{176}{17}\sum_{\substack{(x_1^2+x_2^2+4x_3^2)=3n\\ x_1\equiv x_2\equiv 1 \,(\mathrm{mod}\ 6),\ x_3\equiv 1 \,(\mathrm{mod}\ 3)}}x_1x_2x_3\quad \textit{if}\quad n\equiv 2\ (\mathrm{mod}\ 4),\\
=&\frac{768\cdot u^{\frac{7}{2}}}{17\pi^4r_1^7}L(4,\omega)\sum_{d|r_1}d^7\prod_{p|d}\Big(1-\Big(\frac{\omega}{p}\Big)p^{-4}\Big)\\
&+\frac{276}{17}\sum_{\substack{4(x_1^2+x_2^2)+x_3^2=3n\\ x_1\equiv x_2\equiv 1 \,(\mathrm{mod}\ 3),\ x_3\equiv 1 \,(\mathrm{mod}\ 6)}}x_1x_2x_3\quad \textit{if}\quad n\equiv 3\ (\mathrm{mod}\ 4).
\end{aligned}$$

The values of χ_2 can be calculated by formulas from [5], the values of $L(4,\omega)$ by [3].

In the similar way we can consider the quadratic form $f=x_1^2+x_2^2+x_3^2+4(x_1^2+\cdots+x_9^2)$.

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