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ON THE SPACES OF GENERALIZED THETA-SERIES WITH QUADRATIC FORMS OF FIVE VARIABLES

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Abstract. The spherical polynomials of order $\nu = 2$ with respect to the nondiagonal quadratic form of five variables are constructed and the basis of the space of these spherical polynomials is established. The space of generalized theta-series with respect to the quadratic form of five variables is considered. The basis of this space is constructed.

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1 Introduction.

Let

$$Q(X) = Q(x_1, x_2, \dots, x_r) = \sum_{1\le i\le j\le r} b_{ij} x_i x_j$$

be an integer positive definite quadratic form of r variables and let $A = (a_{ij})$ be the symmetric $r \times r$ matrix of the quadratic form $Q(X)$, where $a_{ii} = 2b_{ii}$ and $a_{ij} = a_{ji} = b_{ij}$, for $i < j$. If $X = (x_1 \dots x_r)^T$ denotes a column matrix and X^T is its transpose, then $Q(X) = \frac{1}{2} X^T A X$. Let A_{ij} denote the cofactor to the element a_{ij} in A and a^*_{ij} is the element of the inverse matrix A^{-1}.

A homogeneous polynomial $P(X) = P(x_1, \cdots, x_r)$ of degree ν with complex coefficients, satisfying the condition

$$\sum_{1\le i,j\le r} a^*_{ij} \Big(\frac{\partial^2 P}{\partial x_i \partial x_j}\Big) = 0 \tag{1}$$

is called a spherical polynomial of order ν with respect to $Q(X)$ (see [1]), and

$$\vartheta(\tau, P, Q) = \sum_{n\in\mathbb{Z}^r} P(n) z^{Q(n)}, \qquad z = e^{2\pi i\tau}, \qquad \tau \in \mathbb{C}, \qquad \operatorname{Im}\tau > 0$$

is the corresponding generalized r-fold theta-series.

Let $P(\nu, Q)$ denote the vector space over $\mathbb{C}$ of spherical polynomials $P(X)$ of even order ν with respect to $Q(X)$. Hecke [2] calculated the dimension of the space $P(\nu, Q)$, $\dim P(\nu, Q) = \binom{\nu+r-1}{r-1} - \binom{\nu+r-3}{r-1}$ and form the basis of the space of spherical polynomials of second order with respect to $Q(X)$.

Let $T(\nu, Q)$ denote the vector space over $\mathbb{C}$ of generalized multiple theta-series, i.e.,

$$T(\nu, Q) = \{\vartheta(\tau, P, Q) \;:\; P \in \mathcal{P}(\nu, Q)\}.$$

Gooding [1] calculated the dimension of the vector space $T(\nu, Q)$ for reduced binary quadratic forms Q. Gaigalas [3] gets the upper bounds for the dimension of the space $T(4, Q)$ and $T(6, Q)$ for some diagonal quadratic forms. In [4-6] we established the upper bounds for the dimension of the space $T(\nu, Q)$ for some quadratic forms of r variables, when $r = 3, 4, 5$, in a number of cases we calculated the dimension and form the bases of these spaces.

In this paper we form the basis of the space of spherical polynomials of second order $P(2, Q)$ with respect to some nondiagonal quadratic form $Q(X)$ of five variables and obtained the basis of the space of generalized theta-series $T(2, Q)$ with spherical polynomial P of second order and nondiagonal quadratic form Q of five variables.

2 The basis of the space $P(2, Q)$ and $T(2, Q)$.

Let

$$Q(X) = b_{11}x_1^2 + b_{22}x_2^2 + b_{33}x_3^2 + b_{44}(x_4^2 + x_5^2) + b_{12}x_1x_2,$$

where $0 < |b_{12}| < b_{11} < b_{22} < b_{33} < b_{44} = b_{55}$ is a nondiagonal quadratic form of five variables. For these forms (see [6])

$$\dim T(\nu, Q) \leq \binom{\frac{\nu}{2} + 3}{3}.$$

We now construct the basis of the space $T(\nu, Q)$, when $\nu = 2$. For the quadratic form $Q(X)$ we have

$$|A| = \det A = 2^3(4b_{11}b_{22} - b_{12}^2)b_{33}b_{44}^2, \qquad a_{11}^* = \frac{2b_{22}}{4b_{11}b_{22} - b_{12}^2},$$

$$a_{12}^* = a_{21}^* = -\frac{b_{12}}{4b_{11}b_{22} - b_{12}^2}, \qquad a_{22}^* = \frac{2b_{11}}{4b_{11}b_{22} - b_{12}^2}, \quad a_{33}^* = \frac{1}{2b_{33}},$$

$$a_{44}^* = a_{44}^* = \frac{1}{2b_{44}}, \quad \text{and other} \quad a_{ij}^* = 0 \ \text{for} \quad i \neq j.$$

Let

$$P(X) = P(x_1, x_2, x_3, x_4, x_5) = \sum_{k=0}^{\nu} \sum_{i=0}^{k} \sum_{j=0}^{i} \sum_{l=0}^{j} a_{kijl} x_1^{\nu-k} x_2^{k-i} x_3^{i-j} x_4^{j-l} x_5^{l}$$

be a spherical function of order ν with respect to the positive quadratic form $Q(x_1, x_2, x_3, x_4, x_5)$ of five variables and let

$$L = [a_{0000}, a_{1000}, a_{1100}, a_{1110}, a_{1111}, a_{2000}, \ldots, a_{\nu\nu\nu\nu}]^T$$

be the column vector, where a_{kijl} $(0 \leq l \leq j \leq i \leq k \leq \nu)$ are the coefficients of the polynomial $P(X)$.

We have (see [5]) that $\dim P(\nu, Q) = \binom{\nu+4}{4} - \binom{\nu+2}{4}$. The polynomials

$$P_{\nu-1,000}(a_{0000}, a_{1000}, \dots, a_{\nu-2,\nu-2,\nu-2,\nu-2}, 1, 0, 0, \dots, 0),$$

$$P_{\nu-1,100}(a_{0000}, a_{1000}, \dots, a_{\nu-2,\nu-2,\nu-2,\nu-2}, 0, 1, 0, \dots, 0),$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$P_{\nu\nu\nu\nu}(a_{0000}, a_{1000}, \dots, a_{\nu-2,\nu-2,\nu-2,\nu-2}, 0, 0, 0, \dots, 1),$$

where the first $\binom{\nu+2}{4}$ coefficients from a_{0000} to $a_{\nu-2,\nu-2,\nu-2,\nu-2}$ are calculated through other $\binom{\nu+4}{4}-\binom{\nu+2}{4}$ coefficients, form the basis of the space $\mathcal{P}(\nu, Q)$ (the coefficients of polynomial P_{bcde} are given in the brackets, a_{bcde} is equal to 1 and the rest of those coefficients for which b is $\nu - 1$ or ν are equal to 0). It is easy to verify, that the spherical polynomials of second order:

$$P_{1000} = \frac{b_{12}}{2b_{22}}x_1^2 + x_1x_2, \qquad P_{1100} = x_1x_3, \qquad P_{1110} = x_1x_4, \qquad P_{1111} = x_1x_5,$$

$$P_{2000} = -\frac{b_{11}}{b_{22}}x_1^2 + x_2^2, \qquad P_{2100} = x_2x_3, \qquad P_{2110} = x_2x_4, \qquad P_{2111} = x_2x_5,$$

$$P_{2200} = -\frac{4b_{11}b_{22} - b_{12}^2}{4b_{22}b_{33}}x_1^2 + x_3^2, \qquad P_{2210} = x_3x_4, \qquad P_{2211} = x_3x_5,$$

$$P_{2220} = -\frac{4b_{11}b_{22} - b_{12}^2}{4b_{22}b_{44}}x_1^2 + x_4^2, \qquad P_{2221} = x_4x_5, \qquad P_{2222} = -\frac{4b_{11}b_{22} - b_{12}^2}{4b_{22}b_{44}}x_1^2 + x_5^2$$

form the basis of the space of spherical polynomials of second order with respect to $Q(x)$.

Now we construct the corresponding generalized theta-series. Consider all possible polynomials P_{kijl}, with even indices i, j, l and $k = \nu - 1, \nu$;

$$\vartheta(\tau, P_{1000}, Q) = \sum_{n=1}^{\infty}\Big(\sum_{Q(x)=n} P_{1000}(x)\Big)z^n = \sum_{n=1}^{\infty}\Big(\sum_{Q(x)=n}\big(\frac{b_{12}}{2b_{22}}x_1^2 + x_1x_2\big)\Big)z^n$$

$$= \frac{b_{12}}{b_{22}}z^{b_{11}} + \dots + 0z^{b_{22}} + \dots + 0z^{b_{33}} + \dots + 0z^{b_{44}} + \cdots,$$

$$\vartheta(\tau, P_{2000}, Q) = \sum_{n=1}^{\infty}\Big(\sum_{Q(x)=n} P_{2000}(x)\Big)z^n = \sum_{n=1}^{\infty}\Big(\sum_{Q(x)=n}\big(-\frac{b_{11}}{b_{22}}x_1^2 + x_2^2\big)\Big)z^n$$

$$= -\frac{2b_{11}}{b_{22}}z^{b_{11}} + \dots + 2z^{b_{22}} + \dots + 0z^{b_{33}} + \dots + 0z^{b_{44}} + \cdots,$$

$$\vartheta(\tau, P_{2200}, Q) = \sum_{n=1}^{\infty}\Big(\sum_{Q(x)=n} P_{2200}(x)\Big)z^n = \sum_{n=1}^{\infty}\Big(\sum_{Q(x)=n}\big(-\frac{4b_{11}b_{22} - b_{12}^2}{4b_{22}b_{33}}x_1^2 + x_3^2\big)\Big)z^n$$

$$= -\frac{4b_{11}b_{22} - b_{12}^2}{2b_{22}b_{33}}z^{b_{11}} + \dots + 0z^{b_{22}} + \dots + 2z^{b_{33}} + \dots + 0z^{b_{44}} + \cdots,$$

$$\vartheta(\tau, P_{2220}, Q) = \sum_{n=1}^{\infty} \Big(\sum_{Q(x)=n} P_{2220}(x) \Big) z^n = \sum_{n=1}^{\infty} \Big(\sum_{Q(x)=n} (-\frac{4b_{11}b_{22} - b_{12}^2}{4b_{22}b_{44}} x_1^2 + x_4^2) \Big) z^n$$

$$= -\frac{4b_{11}b_{22} - b_{12}^2}{2b_{22}b_{44}} z^{b_{11}} + \cdots + 0z^{b_{22}} + \cdots + 0z^{b_{33}} + \cdots + 4z^{b_{44}} + \cdots .$$

These generalized theta-series are linearly independent since the determinant constructed from the coefficients of these theta-series is not equal to zero. We have $\dim T(2, Q) \leq 4$. Hence these theta-series form the basis of the space $T(2, Q)$. We have the following

Theorem. *Let $Q(X)$ be the nondiagonal quadratic form of five variables, given by $Q(X) = b_{11}x_1^2 + b_{22}x_2^2 + b_{33}x_3^2 + b_{44}(x_4^2 + x_5^2) + b_{12}x_1x_2$, then $\dim T(2, Q) = 4$ and the generalized theta-series with spherical polynomials P_{kijl} ($k = 1$ or 2; i, j, l are even):*

$$\vartheta(\tau, P_{1000}, Q); \vartheta(\tau, P_{2000}, Q); \vartheta(\tau, P_{2200}, Q); \vartheta(\tau, P_{2220}, Q)$$

form the basis of the space $T(2, Q)$.

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