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AN APPROXIMATE SOLUTION OF THE ANTI-PLANE PROBLEMS OF THE ELASTICITY THEORY FOR ISOTROPIC COMPOSITE PLANE WEAKENED BY CRACK USING A METHOD OF DISCRETE SINGULARITY

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Abstract. The anti-plane problem of the elasticity theory for a composite (piecewise homogeneous) orthotropic (in particular, isotropic) plane weakened by a crack is reduced to a singular integral equation containing a fixed-singularity with respect to characteristic function of disclosure of crack when the crack reaches the dividing border of interface with the right angle. The method of discrete singularity is applied to finding a solution of the obtained singular integral equation. The corresponding new algorithm is constructed and realized. In this work, behavior of solutions in the neighborhood of the crack endpoints is studied by a method of discrete singularity with uniform division of an interval by knots. The results of computations are represented.

Keywords and phrases: Singular integral equations, crack, anti-plane problem, collocation method, method of discrete singularity, numerical realization.

AMS subject classification (2010): 65R20, 45E05, 45F15, 65M70.

1 Introduction.

Study of boundary value problems for the composite bodies weakened by cracks has a great practical significance. The anti-plane problem of the elasticity theory for a composite (piece-wise homogeneous) orthotropic (in particular, isotropic) plane weakened by a crack, when the crack intersects an interface or reaches this one with the right angle, is studied by the integral equation method. When the crack reaches the interface, the problem is changed with a singular integral equation containing a fixed-singularity, but when the crack intersects the dividing border of interface system (pair) of singular integral equations containing a fixed-singularity concerning characteristic functions of disclosure of crack. The behaviour of the solutions is studied (see [1], [2]). In the article [3] and the present paper develops new computational algorithms for the approximate solution of the above-mentioned problems by the collocation (in particular by a discrete singularity) method ([4]).The algorithms are carried out in various specific practical problems. Numerical results are presented. In the case of loads of different quantities on the crack, the stress intensity factors at the ends of the crack are calculated, which allows us to make a hypothetical prediction about the crack spread.

2 Statement of the problem.

When one half-plane has a rectilinear cut of finite length, which is perpendicular to the boundary and one end of which is located on the boundary, we have one singular integral equation containing a fixed singularity ([1]-[2])

$$\int_0^1 \left(\frac{1}{t-x} - \frac{a_1}{t+x}\right) \varrho_1(t)dt = 2\pi f_1(x), \quad x \in [0;1], \tag{1}$$

where $\varrho_1(x)$, $f_1(x)$ are unknown and given real functions, constant $a_1 = \dfrac{1-\gamma_1}{1+\gamma_1}$, $\gamma_1 = \dfrac{b_{55}^{(1)}}{b_{55}^{(2)}}$, function $f_1(x) = \dfrac{1}{\sqrt{b_{44}^{(1)} \cdot b_{55}^{(1)}}} q_1(x)$; $q_1(x) \in H$, $\varrho_1(x) \in H^*$. $b_{44}^{(k)}$, $b_{55}^{(k)}$ are elastic constants, $k = 1,2$; In particular, if we have an isotropic case, then $b_{44}^{(k)} = b_{55}^{(k)} = \mu_k$, where μ_k is module of displacement, $k = 1,2$. Explanation of behavior of solutions near the ends of the boundary presents a special interest.We get that an order of peculiarity in the point $t = 0$ depends on elastic constants of material and belongs to interval $(0;1)$. $\alpha = 1 - \dfrac{1}{\pi} \arccos\left(\dfrac{b_{55}^{(1)} - b_{55}^{(2)}}{b_{55}^{(1)} + b_{55}^{(2)}}\right) \in (0;1)$, $\beta = \dfrac{1}{2}$. If $b_{55}^{(1)} = b_{55}^{(2)}$, then $\alpha = \frac{1}{2}$. (see [1], [2]). As at the both sides of the body we have infinite solutions, for the existence of uniqueness solution of equation (1) it is necessary to have the following additional condition ([2])

$$\int_0^1 \varrho_1(t)dt = C, \tag{2}$$

where $C = const$ is to be chosen.

3 Algorithm. Now let's consider the singular integral equation (1) containing a fixed singularity. The singular integral equation (1) is solved by a collocation method, in particular, a discrete singular method (see[4]). Let us first of all consider an algorithm of uniform division. Solutions of equation (1) have such a view $\varrho_1(t) = \dfrac{\varrho_1^*(t)}{t^\alpha\sqrt{1-t}}$, $0 < \alpha < 1$, (see [1], [2]). Let's enter such distribution of knots for variables of integration and account points accordingly: $t_{1i} = ih$, $i = 1,2,\cdots,n$; $x_{1j} = t_{1j} + \dfrac{h}{2}$, $j = 1,2,\cdots,n$; $h = \dfrac{1}{n+1}$. The pair of equations (1) and (2) probably to present as follows with the help of quadrature formulas (see [4])

$$\sum_{i=1}^{n} \left(\frac{h}{t_{1i} - x_{1j}} - \frac{a_1 h}{t_{1i} + x_{1j}}\right) \varrho_1(t_{1i}) = 2\pi f_1(x_{1j}), \quad j = 1,2,\cdots,n-1, \tag{3}$$

$$\sum_{i=1}^{n} h\varrho_1(t_{1i}) = C. \tag{4}$$

Thus, we have n equations with n unknowns. It is possible to solve the received system of the linear algebraic equations (3), (4) with the help of one of direct methods, for example, by the Gauss modified method.

4 The numerical realization. For approximate solution of singular integral equation (1) the several programs in "Maple" is composed. The algorithm has been approved tests and the results of numerical calculations are represented in tables. In the above-mentioned research tasks our main objective was research of a possible distribution of cracks along a body, study behavior of solution (the character function of stress) and finding the coefficients of intensity of stress (stress intensity factor sif_1, sif_2) in a vicinity at the ends of the cracks. As we have mentioned a main objective was studying of behavior of solution in a vicinity at the ends of cracks and a finding the coefficients of intensity of stress. For this purpose we have calculated values of the coefficients of intensity of stress sif_1, sif_2: $\mathbf{sif_1} = \lim\limits_{\mathbf{x}\to\mathbf{0}+} \mathbf{x}^{\alpha}\boldsymbol{\varrho}_\mathbf{1}(\mathbf{x})$, $\mathbf{sif_2} = \lim\limits_{\mathbf{x}\to\mathbf{1}-} \sqrt{\mathbf{1-x}}\boldsymbol{\varrho}_\mathbf{1}(\mathbf{x})$, $sif_1 \approx x_{11}^{\alpha}\varrho_1(x_{11})$, $sif_2 \approx \sqrt{1-x_{1n}}\varrho_1(x_{1n})$, at the ends of cracks by using algorithm of interval $[0,+1]$ uniform splitting and on each step of calculations increase number of division of an interval two times. The approached values of coefficients of intensity of stresses (stress intensity factor) in the vicinity of the ends of cracks. Consider a body composed of two isotropic materials (copper and aluminum).

Consider two cases: problem 1 (aluminum copper, copper has a crack). For copper $b_{44}^{(1)} = b_{55}^{(1)} = 45.5GPa$ and for aluminum $b_{44}^{(2)} = b_{55}^{(2)} = 25.5GPa$. $\alpha = 1 - \frac{1}{\pi}\arccos\left(\frac{20}{71}\right) \approx 0.59$. Numerical calculations for the functions $q_1(x)$ we are studying the following two cases: 1. $q_1(x) = 0.01GPa,\ C = 0$, (variant 1), 2. $q_1(x) = 0.02GPa,\ C = 0$, (variant 2), are given by the Table 1

variant	$sif \setminus n$	8	16	32	64	128
1	sif_1	-0.000059	-0.000062	-0.000064	-0.000064	-0.000065
	sif_2	0.000039	0.000042	0.000044	0.000045	0.000045
2	sif_1	-0.000118	-0.000125	-0.000127	-0.000129	-0.000129
	sif_2	0.000078	0.000084	0.000088	0.000089	0.000090

Table 1.

problem 2 (copper aluminum, aluminum has a crack). For copper $b_{44}^{(2)} = b_{55}^{(2)} = 45.5GPa$ and for aluminum $b_{44}^{(2)} = b_{55}^{(2)} = 25.5GPa$. $\alpha = 1 - \frac{1}{\pi}\arccos\left(\frac{20}{71}\right) \approx 0.41$. Numerical calculations for the functions $q_1(x)$ we are studying the following two cases: 1. $q_1(x) = 0.01GPa, C = 0$ (variant 3), 2. $q_1(x) = 0.02GPa, C = 0$ (variant 4), are given by Table 2

variant	$sif \setminus n$	8	16	32	64	128
3	sif_1	-0.000131	-0.000144	-0.000151	-0.000155	-0.000158
	sif_2	0.000066	0.000071	0.000073	0.000074	0.000075
4	sif_1	-0.000262	-0.000288	-0.000302	-0.000311	-0.000316
	sif_2	0.000132	0.000142	0.000146	0.000149	0.000150

Table 2.

Remark. In a task with physical content, the number C modulus must be very small ($C \approx 0$). If the absolute values of the coefficients of intensity of tension are less than 1 (it to very close to critical limit of distribution of a crack) but it is close to unit then cracks develop slowly. If the absolute values of the coefficients of intensity of tension are considerably less than 1, then cracks almost does't develop. Numerical experiments have shown that increment of loading at the ends of the crack causes increment of values of the coefficients of the intensity of tension. As we consider linear tasks of the elasticity theory, increment or diminution loading will lead to proportionally increment or diminution of values of relevant solutions. The last fact gives possibility to make the hypothetical forecasts about developments of a crack.

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