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THE BENDING PROBLEM OF AN INFINITE ANISOTROPIC PLATE WITH A CIRCULAR HOLE AND CUTS

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Abstract. The anisotropic plate, weakened with a circular hole and two symmetrical cuts is considered. The bending moment acts on the plate at infinity and the boundary of the hole is rigidly fixed. On the boundary of the cuts the jumps of the plate bending, of the angle of rotation, of the bending moment and of the lateral force are given. Using the methods of the theory of analytic functions the problem is reduced to a Riemann problem on the segment and to the boundary value problem for a circle. The solution of the problem is presented in an explicit form.

Keywords and phrases: The anisotropic plate, the bending problem, the boundary value problems, effective solution.

AMS subject classification (2010): 74B05, 74K20, 74S70, 65L10.

1 Introduction. The analytical solutions of problems of bending of a finite or infinite plate with various boundary conditions have been investigated [1-5]. The effective and approximate solutions of contact and mixed boundary value problems of bending of plates weakened with defects or reinforced with elastic(rigid) thin-walled elements are obtained, and the behavior of contact stresses at the ends of the contact line is established [6-11]. As is well known the bending equation of anisotropic plate has the following form

$$D_{11}\frac{\partial^4 W}{\partial x^4}+4D_{16}\frac{\partial^4 W}{\partial x^3\partial y}+2(D_{12}+2D_{66})\frac{\partial^4 W}{\partial x^2\partial y^2}+4D_{26}\frac{\partial^4 W}{\partial x\partial y^3}+D_{22}\frac{\partial^4 W}{\partial y^4}=q, \tag{1}$$

where q is the load acting on the unit of area, constants D_{ij} are rigidity of an anisotropic plate. The solution of equation (1) depends on the roots μ_1, μ_2, $\overline{\mu}_1$, $\overline{\mu}_2$ of corresponding characteristic equation. The solution of the equation (1) has the following forms [1]

$$W=W_0+2\mathrm{Re}\left[W_1(z_1)+W_2(z_2)\right], \quad \mu_1\neq\mu_2 \tag{2}$$

where W_0 is solution of nonhomogeneous equation (1), W_1 and W_2 are analytic functions of complex variables $z_1=x+\mu_1 y$ and $z_2=x+\mu_2 y$, respectively.

2 Statement and solution of the problem. An infinite anisotropic plate weakened by anunit radius hole and symmetrically arranged cuts is considered. The cuts are placed along the sections $y=0$, $b<x<c$, $-c<x<-b$, $(b>1)$. The boundary of the hole is rigidly fixed, the bending moments and generalized lateral forces $M_x^\infty=M$,

$M_y^\infty = 0$, $N_x^\infty = 0$, $N_y^\infty = 0$ acts on the plate at infinity. In the plate the bending function, bending moments, torque and cutting force have to be determined. The problem is equivalent to finding a solution to a homogeneous equation (1), with boundary value conditions on the hole

$$W = 0, \quad \frac{\partial W}{\partial n} = 0, \quad (x,y) \in \gamma \tag{3}$$

and following conditions on the cuts [6]

$$\begin{gathered} < W >= 0, \ < W_y' >= 0, \ < M_y >= 0, \ < N_y >= \mu(x), \\ y = 0, \ x \in l \equiv (-c,-b) \cup (b,c), \end{gathered} \tag{4}$$

where γ is the boundary of the unit radius circle, $< f >= f(x,0^-) - f(x,0^-+$. W is bending function of plate, W_y', M_y and N_y are angle of rotation, the bending moment and the lateral force in the plate, respectively. $\mu(x)$ is a known function from the Holder's (H) class [13].

Let the anisotropic plate occupies the area S of the complex plane, then the functions $W_1(z_1)$ and $W_2(z_2)$ will be defined, respectively in the areas S_1 and S_2, which are obtained from S by the following affine transformations

$$x_1 = x + \alpha y, y_1 = \beta y \text{ and } x_2 = x + \gamma y, y_2 = \delta y$$

where $\alpha + i\beta = \mu_1$ and $\gamma + i\delta = \mu_2$, $\mu_1 \neq \mu_2$, then z_1 and z_2 complex variables are represented as follows

$$z_1 = x + \mu_1 y = \frac{1 - i\mu_1}{2}\left(z + \frac{1+\mu_1}{1-i\mu_1}\bar{z}\right), \quad z_2 = x + \mu_2 y = \frac{1 - i\mu_2}{2}\left(z + \frac{1+\mu_2}{1-i\mu_2}\bar{z}\right).$$

From formula (2) and conditions (4) we obtain the following relations

$$\begin{aligned} &< W_1'(x) + W_2'(x) + \overline{W_1'(x)} + \overline{W_2'(x)} >= 0, \\ &< \mu_1 W_1'(x) + \mu_2 W_2'(x) + \overline{\mu_1}\overline{W_1'(x)} + \overline{\mu_2}\overline{W_2'(x)} >= 0, \\ &< q_1 W_1''(x) + q_2 W_2''(x) + \overline{q_1}\overline{W_1''(x)} + \overline{q_2}\overline{W_2''(x)} >= 0, \\ &< s_1 W_1'''(x) + s_2 W_2'''(x) + \overline{s_1}\overline{W_1'''(x)} + \overline{s_2}\overline{W_2'''(x)} >= \mu(x). \end{aligned}$$

By solving the system of algebraic equations with respect to jumps $< W_1'''(x) >$, $< W_2'''(x) >$, $< \overline{W_1'''(x)} >$, $< \overline{W_2'''(x)} >$ we get the following boundary value problems of linear conjugation [12]

$$[W_1'''(x)]^- - [W_1'''(x)]^+ = \tfrac{\Delta_1}{\Delta}\mu(x), \ \ [W_2'''(x)]^- - [W_2'''(x)]^+ = \tfrac{\Delta_2}{\Delta}\mu(x), \ \ x \in l \tag{5}$$

where

$$\Delta = \begin{vmatrix} 1 & 1 & 1 & 1 \\ \mu_1 & \mu_2 & \overline{\mu_1} & \overline{\mu_2} \\ q_1 & q_2 & \overline{q_1} & \overline{q_2} \\ s_1 & s_2 & \overline{s_1} & \overline{s_2} \end{vmatrix} \neq 0, \quad \Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ \mu_2 & \overline{\mu_1} & \overline{\mu_2} \\ q_2 & \overline{q_1} & \overline{q_2} \end{vmatrix} \neq 0, \quad \Delta_2 = \begin{vmatrix} 1 & 1 & 1 \\ \mu_1 & \overline{\mu_1} & \overline{\mu_2} \\ q_1 & \overline{q_1} & \overline{q_2} \end{vmatrix} \neq 0.$$

Since $W_1'''(\infty) = 0$ and $W_2'''(\infty) = 0$ the solutions to problems (5) are represented as follows

$$
\begin{aligned}
W_1'(z_1) &= \tfrac{\Delta_1}{2\pi i\Delta}\int_l (t-z_1)\ln(t-z_1)\mu(t)dt + Bz_1 + F_1(z_1),\\
W_2'(z_2) &= \tfrac{-\Delta_2}{2\pi i\Delta}\int_l (t-z_2)\ln(t-z_2)\mu(t)dt + B_1z_2 + F_2(z_2),
\end{aligned} \tag{6}
$$

where $F_1(z_1)$ and $F_2(z_2)$ are unknown analytic functions in S_1 and S_2 areas, respectively. Based to the relation (2) and condition (3) we have the boundary value problems on γ

$$
\begin{gathered}
F_1^0(\sigma) + F_2^0(\sigma) + \overline{F_1^0(\sigma)} + \overline{F_2^0(\sigma)} = g_1(\sigma),\\
\mu_1 F_1^0(\sigma) + \mu_2 F_2^0(\sigma) + \overline{\mu_1}\,\overline{F_1^0(\sigma)} + \overline{\mu_2}\,\overline{F_2^0(\sigma)} = g_2(\sigma), \quad |\sigma| = 1
\end{gathered}
$$

where

$$
\begin{aligned}
&F_k^0(\sigma) \equiv F_k\left(R_k\left(\sigma + \tfrac{m_k}{\sigma}\right)\right),\ f_k^0(\sigma) \equiv f_k\left(R_k\left(\sigma + \tfrac{m_k}{\sigma}\right)\right),\ R_k = \tfrac{1-i\mu_k}{2},\ m_k = \tfrac{1+\mu_k}{1-i\mu_k},\\
&f_1(z_1) = \tfrac{\Delta_1}{2\pi i\Delta}\int_l (t-z_1)\ln(t-z_1)\mu(t)dt,\ f_2(z_2) = \tfrac{\Delta_2}{2\pi i\Delta}\int_l (t-z_2)\ln(t-z_2)\mu(t)dt,\\
&g_1(\sigma) = -f_1^0(\sigma) - f_2^0(\sigma) - \overline{f_1^0(\sigma)} - \overline{f_2^0(\sigma)} - BR_1\left(\sigma + \tfrac{m_1}{\sigma}\right) - B_1R_2\left(\sigma + \tfrac{m_2}{\sigma}\right)\\
&-B\overline{R_1}\left(\tfrac{1}{\sigma} + \overline{m_1}\sigma\right) - B_1\overline{R_2}\left(\tfrac{1}{\sigma} + \overline{m_2}\sigma\right),\\
&g_2(\sigma) = -\mu_1 f_1^0(\sigma) - \mu_2 f_2^0(\sigma) - \overline{\mu_1}\,\overline{f_1^0(\sigma)} - \overline{\mu_2}\,\overline{f_2^0(\sigma)} - \mu_1 BR_1\left(\sigma + \tfrac{m_1}{\sigma}\right)\\
&-\mu_2 B_1R_2\left(\sigma + \tfrac{m_2}{\sigma}\right) - \overline{\mu_1}B\overline{R_1}\left(\tfrac{1}{\sigma} + \overline{m_1}\sigma\right) - \overline{\mu_2}B_1\overline{R_2}\left(\tfrac{1}{\sigma} + \overline{m_2}\sigma\right),\quad k = 1,2
\end{aligned}
$$

Using the methods of analytic functions theory and the theory of Cauchy-type integral we obtain the following equations [12]

$$
\begin{aligned}
-F_1^0(\zeta) - F_2^0(\zeta) &= \overline{f_1^0}\left(\tfrac{1}{\zeta}\right) + \overline{f_2^0}\left(\tfrac{1}{\zeta}\right) + \tfrac{\Gamma_1}{\zeta},\quad \zeta \in S\\
-\mu_1 F_1^0(\zeta) - \mu_2 F_2^0(\zeta) &= \overline{\mu_1}\,\overline{f_1^0}\left(\tfrac{1}{\zeta}\right) + \overline{\mu_2}\,\overline{f_2^0}\left(\tfrac{1}{\zeta}\right) + \tfrac{\Gamma_1}{\zeta},\quad \zeta \in S
\end{aligned} \tag{7}
$$

where

$$
\begin{aligned}
&\Gamma_1 = BR_1m_1 + B_1R_2m_2 + B\overline{R_1} + \overline{B_1R_2}, && (p_1 + \overline{p_1})B + p_2B_1 + \overline{p_2}\,\overline{B_1} = -M,\\
&\Gamma_2 = \mu_1 BR_1m_1 + \mu_2 B_1R_2m_2 + \overline{\mu_1}B\overline{R_1} + \overline{\mu_2}\,\overline{B_1R_2}, && (q_1 + \overline{q_1})B + q_2B_1 + \overline{q_2}\,\overline{B_1} = 0,\\
& && (r_1 + \overline{r_1})B + r_2B_1 + \overline{r_2}\,\overline{B_1} = 0.
\end{aligned}
$$

By solving of system (7) we obtain

$$
F_1^0(\zeta) = -\overline{f_1^0}\left(\frac{1}{\zeta}\right) + \frac{\Gamma_2 - \mu_2\Gamma_1}{\mu_2 - \mu_1}\frac{1}{\zeta},\quad F_2^0(\zeta) = -\overline{f_2^0}\left(\frac{1}{\zeta}\right) + \frac{\mu_1\Gamma_1 - \Gamma_2}{\mu_2 - \mu_1}\frac{1}{\zeta},
$$

from which the functions $F_1(z_1)$ and $F_2(z_2)$ are represented by the formulas

$$
\begin{aligned}
&F_1(z_1) = \tfrac{\overline{\Delta_1}}{2\pi i}\int_l\left(t - \tfrac{2R_1}{z_1+\sqrt{z_1^2-4R_1^2m_1}}\right)\ln\left(t - \tfrac{2R_1}{z_1+\sqrt{z_1^2-4R_1^2m_1}}\right)\mu(t)dt\\
&+\tfrac{\Gamma_2-\mu_2\Gamma_1}{\mu_2-\mu_1}\tfrac{2R_1}{z_1+\sqrt{z_1^2-4R_1^2m_1}},\\
&F_2(z_2) = \tfrac{\overline{\Delta_2}}{2\pi i}\int_l\left(t - \tfrac{2R_1}{z_2+\sqrt{z_2^2-4R_2^2m_2}}\right)\ln\left(t - \tfrac{2R_1}{z_2+\sqrt{z_2^2-4R_2^2m_2}}\right)\mu(t)dt\\
&+\tfrac{\mu_1\Gamma_1-\Gamma_2}{\mu_2-\mu_1}\tfrac{2R_1}{z_2+\sqrt{z_2^2-4R_2^2m_2}}.
\end{aligned} \tag{8}
$$

3 Conclusion. The solution of the considered problem is represented effectively by formulas (6) and (8).

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